Question 1

(a) Find the Fourier cosine series for the function $f(x) = x^2$ for $0 \le x \le 1$.

Sketch the graph of the function to which the Fourier series converges for the range $-3 \le x \le 3$.

Find the sum to infinity of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ **[11 marks]**

(b) By using the Fourier Inversion Theorem find the function $f(x)$ whose Fourier transform is $\hat{f}(\xi) = e^{-\alpha\xi^2}$ where α is a positive constant.

Hence, by taking Fourier tranforms, show that the solution of the equation

$$u_t = u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0; \quad u(x,0) = g(x) \text{ for } -\infty < x < \infty$$

is

$$u(x,t) = \sqrt{\frac{1}{4\pi t}} \int_{-\infty}^{\infty} g(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds$$

[9 marks]

Question 2

(a) Show that there exist infinitely many positive eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ of the boundary value problem

$$-y'' = \lambda y; \quad y(0) = 0, \; y(1) + 2y'(1) = 0.$$

What is the approximate value of λ_n when n is large? **[10 marks]**

(b) Solve the following PDE

$$x^2 u_x + y^2 u_y = 0; \quad u(1,y) = y.$$

[10 marks]

Question 3

(a) Find the solution of the heat equation $u_t = k u_{xx}$ for $0 < x < L$, $t > 0$ subject to the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ for $t > 0$ and an initial condition

$$u(x,0) = \begin{cases} T_0 \text{ for } 0 \le x \le \frac{L}{2} \\ 0 \text{ for } \frac{L}{2} < x \le L \end{cases}$$

Explain briefly the physical situation represented by the equation above.

[14 marks]

(b) If $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx} \text{ for } 0 \le x \le 1 \text{ and } t > 0$$

the initial condition $u(x,0) = 0$ for $0 \le x \le 1$ and the boundary conditions $u(0,t) = 0 = u(1,t)$ for $t > 0$, show by considering the function $E(t) = \int_0^1 u^2(x,t)\, dx$ that $u(x,t) \equiv 0$.

[6 marks]

continued overleaf

Question 4

(a) Solve $\nabla^2 u(x,y) = 0$ on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$ subject to the boundary conditions

$$u(x,0) = 0 \text{ and } u(x,1) = \sin(\pi x) \text{ for } 0 \leq x \leq 2; \; u(0,y) = u(2,y) = 0 \text{ for } 0 \leq y \leq 1.$$

[10 marks]

(b) The steady state temperature $u(r,\theta)$ of a plate in the shape of the circle centre the origin and radius 1 satisfies the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

where (r,θ) denotes plane polar co-ordinates.
If the steady state solution satisfies the boundary condition that $u(1,\theta) = \cos^2(\theta)$, find $u(r,\theta)$. **[10 marks]**

Question 5

(a) By taking Laplace transforms solve the initial value problem

$$x'(t) = 3x(t) - y(t); \quad y'(t) = 5x(t) - y(t); \quad x(0) = 1; \quad y(0) = 2.$$

[7 marks]

(b) By considering the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the system of equations

$$x'(t) = x(t) - 3y(t); \quad y'(t) = 4x(t) - 6y(t)$$

and hence sketch the phase plane for the system. **[11 marks]**

Write down the solution of the above system satisfying the initial condition
$x(0) = 2$, $y(0) = 2$. **[2 marks]**

Question 6

(a) Determine the nature and stability of the equilibrium point $(0,0)$ for the system

$$x' = 3x - 4y + x^2 - y^2; \quad y' = x - 2y - y^3.$$

[5 marks]

(b) Write the equation

$$x'' = x - x^3$$

as a system of first order equations and find all the equilibrium points and the equations of the trajectories in the phase plane

Hence sketch the phase plane for the system. **[15 marks]**

continued overleaf

Question 7

(a) Find all equilibrium points for the system

$$x' = x(3 - y); \quad y' = y(1 - y + x)$$

and determine the nature of each.

Hence sketch the phase plane of the system in the first quadrant.

Sketch the graphs of the functions $x(t)$ and $y(t)$ which are the solutions of the above system satisfying the initial condition $x(0) = 1$ and $y(0) = 1$.

[14 marks]

(b) By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, determine whether or not $(0, 0)$ is a stable equilibrium point for the system

$$x' = 2y - xy^2; \qquad y' = -x - y^3.$$

[6 marks]

END OF PAPER