Solutions to Mathematical Techniques Exam - June 2001

1. $a_0 = \frac{1}{1}\int_0^1 (1-x)\,dx = x - \frac{1}{2}x^2|_0^1 = \frac{1}{2}$.

$a_n = \frac{2}{1}\int_0^1 (1-x)\cos(n\pi x)\,dx$

$= 2\,[\,\frac{1}{n\pi}(1-x)\sin(n\pi x)\,|_{x=0}^{x=1} + \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\,]$

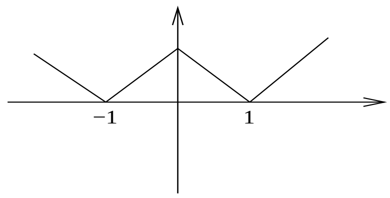
$= -\frac{2}{n^2\pi^2}\,\cos(n\pi x)\,|_0^1 = \frac{2}{n^2\pi^2}[\,1-\cos(n\pi)\,]$

$= \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{4}{n^2\pi^2} \text{ if } n \text{ is odd.} \end{cases}$

Hence Fourier cosine series is

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\,[\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \ldots]$$

Since Fourier cosine series converges to an even function of period 2, the series converges to the following limit on $[-2,2]$



2. (a) $-y'' = \lambda y; \quad 2y(0) + y'(0) = 0, \quad y(1) = 0.$

Suppose $\lambda < 0$; then we may write $\lambda = -k^2$ where $k > 0$.

Then $y'' = k^2 y$ has general solution $y = A\,\cosh(kx) + B\,\sinh(kx)$.

We have $y'(x) = Ak\,\sinh(kx) + Bk\,\cosh(kx)$. Hence

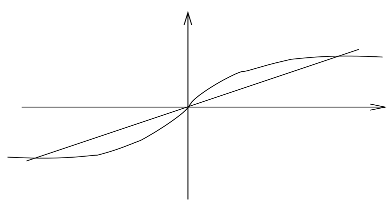
$2y(0) + y'(0) = 0 \Longleftrightarrow 2A + Bk = 0 \Longleftrightarrow B = -\frac{2}{k}A$.

Also $y(1) = 0 \Longleftrightarrow A\,\cosh(k) + B\,\sinh(k) = 0$.

Thus we have a solution provided $A\,\cosh(k) - \frac{2}{k}A\,\sinh(k) = 0$, i.e., provided $A = 0$ or $\tanh(k) = \frac{k}{2}$.

Thus we obtain a non-zero solution provided that $\tanh(k) = \frac{k}{2}$. We can see from the diagram below that there is exactly one value of $k > 0$ such that $\tanh(k) = \frac{k}{2}$.

Hence the equation has a negative eigenvalue.



3. $u_x + 2u_y = u\,; \qquad u(x,0) = 2x.$

Characteristics are solutions of $\frac{dy}{dx} = 2$, i.e., $y = 2x + c$.

Let $(x_0, y_0) \in \mathbf{R^2}$. The characteristic through (x_0, y_0) is $y = 2x + (y_0 - 2x_0)$ and this meets the line $y = 0$ where $x = x_0 - \frac{y_0}{2}$, i.e., at $(x_0 - \frac{y_0}{2}, 0)$.

Let $v(x) = u(x, y(x))$ where $y = y(x)$ is the above characteristic.

Then $v(x_0 - \frac{y_0}{2}) = u(x_0 - \frac{y_0}{2}, 0) = 2x_0 - y_0$.

Also $\frac{dv}{dx} = u_x + u_y \frac{dy}{dx} = u_x + 2u_y = u = v$.

But $\frac{dv}{dx} = v \Longrightarrow v = Ke^x$.

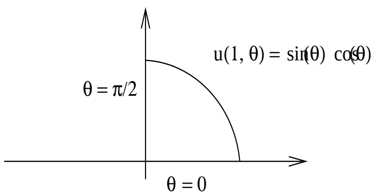
Since $v(x_0 - \frac{y_0}{2}) = 2x_0 - y_0$, $\quad 2x_0 - y_0 = Ke^{x_0 - \frac{y_0}{2}}$ and so $K = (2x_0 - y_0)e^{\frac{y_0}{2} - x_0}$.

Hence $v(x) = Ke^x = (2x_0 - y_0)e^{\frac{y_0}{2} - x_0}\, e^x$.

Thus $u(x_0, y_0) = v(x_0) = (2x_0 - y_0)e^{\frac{y_0}{2}}$.

Hence we have $u(x, y) = (2x - y)e^{\frac{y}{2}}$.

4. $$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$



We seek solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$.

We require $\Theta(0) = 0$ and $\Theta(\frac{\pi}{2}) = 0$.

Also we have a solution to the equation if

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and so

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = k \text{ where } k \text{ is a constant.}$$

Thus we require

$$r^2R'' + rR' - kR = 0 \qquad (1)$$

$$-\Theta'' = k\Theta; \quad \Theta(0) = 0, \quad \Theta(\tfrac{\pi}{2}) = 0 \qquad (2)$$

(2) has non-zero solutions if and only if $k = \frac{n^2\pi^2}{(\pi/2)^2} = 4n^2$ for $n = 1, 2 \ldots$ with corresponding solutions $\sin(2n\theta)$.

When $k = 4n^2$, (1) becomes

$$r^2R'' + rR' - 4n^2R = 0 \quad \text{-} \quad \text{an Euler equation}$$

which has solutions r^{2n} and r^{-2n}.

Since we require solutions to be bounded at $r = 0$, we do not make use of the solutions r^{-2n}.

Thus equation has solutions $r^{2n}\sin(2n\theta)$ for $n = 1, 2, \ldots$.
Thus we have solution

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin(2n\theta)$$

for any choice of the coefficients A_n.
Since $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta) = u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin(2n\theta)$, we choose $A_1 = \frac{1}{2}$ and $A_n = 0$ for $n > 1$.
Hence we have solution $u(r,\theta) = \frac{1}{2}r^2\sin(2\theta)$.

5. $u_{tt} = u_{xx}, \qquad 0 < x < L, \quad t > 0.$
We seek solutions of the form $u(x,t) = X(x)\,T(t)$.
To ensure that $u(0,t) = u(L,t) = 0$, we require that $X(0) = 0 = X(L)$.
To ensure that u is a solution we require $X(x)\,T''(t) = X''(x)\,T(t)$,
i.e., $\frac{X''}{X} = \frac{T''}{T} = -k$ where k is a constant.
Thus we require

$$\begin{aligned} X'' &= -kX; \quad X(0) = 0 = X(L) \qquad (1)\\ T'' &= -kT \qquad\qquad\qquad\qquad\quad (2) \end{aligned}$$

(1) has nonzero solutions if and only if $k = \frac{n^2\pi^2}{L^2}$ with corresponding solutions $\sin(\frac{n\pi x}{L})$ for $n = 1, 2, \ldots$
If $k = \frac{n^2\pi^2}{L^2}$, (2) has general solution $T(t) = A_n\cos(\frac{n\pi t}{L}) + B_n\sin(\frac{n\pi t}{L})$.
Thus we have non-zero solution

$$u(x,t) = \sum_{n=1}^{\infty}\{A_n\cos(\tfrac{n\pi t}{L}) + B_n\sin(\tfrac{n\pi t}{L})\}\sin(\tfrac{n\pi x}{L}).$$

To ensure that $u(x,0) = 0$ we choose $A_n = 0$ for all n.
Hence $u(x,t) = \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi t}{L})\,\sin(\frac{n\pi x}{L})$ and so
$u_t(x,t) = \sum_{n=1}^{\infty}\frac{n\pi}{L}B_n\cos(\frac{n\pi t}{L})\,\sin(\frac{n\pi x}{L})$.
Since $u_t(x,0) = 2$, we require $\sum_{n=1}^{\infty}\frac{n\pi}{L}B_n\sin(\frac{n\pi x}{L}) = 2$.
Hence we choose B_n so that $\frac{n\pi}{L}B_n$ are the coefficients in Fourier sin series
i.e., $\frac{n\pi}{L}B_n = \frac{2}{L}\int_0^L 2\sin(\frac{n\pi x}{L})\,dx = \frac{4}{n\pi}[1 - \cos(n\pi)]$
i.e., we choose $B_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{8L}{n^2\pi^2} \text{ if } n \text{ is odd.} \end{cases}$

6. $(i) \quad x' = 3x - y, \quad y' = 5x - y\ ; \qquad x(0) = 1, \quad y(0) = 2.$
Taking Laplace transforms we obtain

$$\begin{aligned} s\overline{x}(s) - x(0) &= 3\overline{x}(s) - \overline{y}(s); \quad \text{i.e., } (s-3)\,\overline{x}(s) + \overline{y}(s) = 1 \qquad (1)\\ s\overline{y}(s) - y(0) &= 5\overline{x}(s) - \overline{y}(s); \quad \text{i.e., } -5\overline{x}(s) + (s+1)\,\overline{y}(s) = 2 \quad (2) \end{aligned}$$

$5\times$ (1) + $(s-3)\times$(2) gives $[5 + (s+1)(s-3)]\,\overline{y}(s) = 5 + 2(s-3)$,
i.e., $(s^2 - 2s + 2))\,\overline{y}(s) = 2s - 1$,

i.e., $\overline{y}(s) = \frac{2s-1}{(s-1)^2+1} = 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$.
Hence $y(t) = 2e^t\cos(t) + e^t\sin(t)$.

$(s+1)\times$ (1) - (2) gives $[(s-3)(s+1)+5]\,\overline{x}(s) = s+1-2$,
i.e., $\overline{x}(s) = \frac{s-1}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1}$.
Hence $x(t) = e^t\cos(t)$.

7(a) $x' = 2x - 4y, \quad y' = x - 3y.$

The system can be written as $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

λ is an eigenvalue of A if and only if det $\begin{bmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix} = 0$,

i.e., $(2-\lambda)(-3-\lambda)+4 = 0$, i.e., $\lambda^2+\lambda-2 = 0$, i.e., $(\lambda+2)(\lambda-1) = 0$,
i.e., $\lambda = -2,\ 1$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$ if and only if

$$2x - 4y = -2x \qquad \text{i.e., } 4x - 4y = 0.$$
$$x - 3y = -2y \qquad \text{i.e., } x - y = 0.$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$.

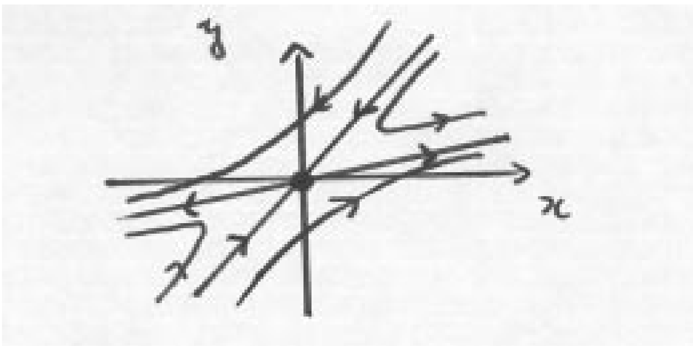
$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 1$ if and only if

$$2x - 4y = x \qquad \text{i.e., } x - 4y = 0$$
$$x - 3y = y \qquad \text{i.e., } x - 4y = 0.$$

Thus $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 1$.

Thus system has general solution $\underline{x}(t) = c_1\,e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\,e^t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

The phase plane is



7(b) $x' = ax + by, \quad y' = cx + dy.$

The system can be written as $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

λ is an eigenvalue of A if and only if det $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$,

i.e., iff $(a-\lambda)(d-\lambda) - bc = 0$, i.e., $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$,

i.e., iff $\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{a+d \pm \sqrt{\Delta}}{2} = \alpha, \beta$(say)

(i) Suppose $\Delta \geq 0$.
Then α and β are real and since $ad - bc > 0$, $(a+d)^2 > \Delta$.
Hence $\alpha, \beta = \frac{a+d \pm \sqrt{\Delta}}{2} < 0$ since $a + d < 0$.
Thus $(0,0)$ is a stable node.

(ii) Suppose $\Delta < 0$.
Then α and β are complex conjugates with negative real part $\frac{a+d}{2}$.
Thus $(0,0)$ is a stable spiral point.

8. $\frac{dx}{dt} = x(1-y); \quad \frac{dy}{dt} = y(3-x)$

Equilibrium points occur when $x = 0, y = 0$ and when $y = 1$ and $x = 3$.

If $f(x,y) = x(1-y)$, $\frac{\partial f}{\partial x}(x,y) = 1 - y$ and $\frac{\partial f}{\partial y}(x,y) = -x$.

If $g(x,y) = y(3-x)$, $\frac{\partial g}{\partial x}(x,y) = -y$ and $\frac{\partial g}{\partial y}(x,y) = 3 - x$.

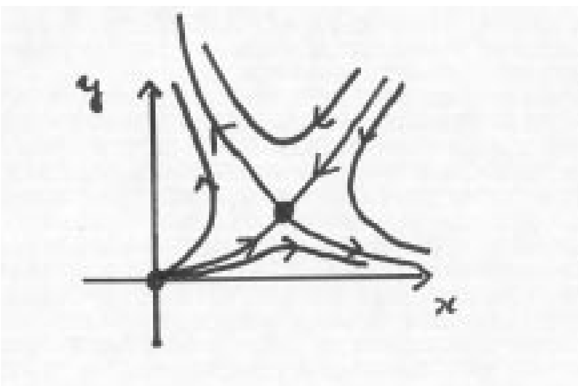
Hence linearized equation at $(0,0)$ is $\underline{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A are $\lambda = 1, 3$ and so $(0,0)$ is an unstable node.

Linearized equation at $(3,1)$ is $\underline{x}' = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A satisfy $\lambda^2 - 3 = 0$, i.e., $\lambda = \pm\sqrt{3}$.
Hence $(3,1)$ is a saddle point.

Phase plane is



9.(i) $x' = y^3 - x^3; \quad y' = -2xy^2.$

Let $V(x,y) = ax^2 + by^2$. Then

$\frac{d}{dt}\left[V(x(t),y(t))\right] = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} = 2ax(y^3 - x^3) + 2by\,.\,(-2xy^2)$

$= -2ax^4 + (2a - 4b)xy^3.$

Choosing $a = 2$, $b = 1$, we obtain $V(x,y) = 2x^2 + y^2$ and

$\frac{d}{dt}[V(x(t),y(t))] = -4x^4 \leq 0.$

Thus $(0,0)$ is a stable equilibrium point.

(ii) $x' = x^3 - y^3; \quad y' = -2xy^2.$

It is clear that the positive x-axis is a trajectory and at all points on this trajectory $y' = 0$ and $x' = x^3$.

Hence for this trajectory $x(t)$ increases as t increases and so the trajectory does not stay close to $(0,0)$ as t increases.

Hence $(0,0)$ is an unstable equilibrium point.