1. Find the Fourier cosine series for the function $f(x) = 1 - x$ valid for $0 < x < 1$.

Sketch the graph of the function to which the series converges over the range $-2 < x < 2$. **[8 marks]**

2. Show that the boundary value problem

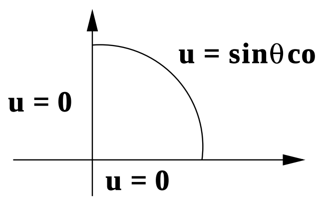
$$-y'' = \lambda y; \quad 2y(0) + y'(0) = 0, \quad y(1) = 0$$

has a negative eigenvalue. **[8 marks]**

3. Solve the following PDE

$$u_x + 2u_y = u; \quad u(x, 0) = 2x.$$ **[9 marks]**

4. The steady state temperature $u(r, \theta)$ of a plate in the shape of the quarter circle $\{(r, \theta) : 0 \le r \le 1 \text{ and } 0 \le \theta \le \frac{\pi}{2}\}$



satisfies the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

where (r, θ) denotes plane polar co-ordinates.

If the steady state solution satisfies the boundary conditions that $u = 0$ on both the x and y-axes axis and $u(1, \theta) = \sin\theta\cos\theta$ for $0 < \theta < \frac{\pi}{2}$, find $u(r, \theta)$. **[13 marks]**

5. A string of length L is fixed at its ends at $x = 0$ and $x = L$ which lie on the same horizontal level. When the string vibrates, its displacement from the horizontal satisfies the equation

$$u_{tt} = u_{xx}\ ; \qquad 0 < x < L, \quad t > 0.$$

If the string has zero initial displacement, i.e., $u(x, 0) = 0$ for $0 \le x \le L$ and has initial velocity $u_t(x, 0) = 2$ for $0 \le x \le L$, determine $u(x, t)$. **[16 marks]**

continued overleaf

6. By taking Laplace transforms, solve the initial value problem

$$x'(t) = 3x(t) - y(t); \quad y'(t) = 5x(t) - y(t); \quad x(0) = 1; \; y(0) = 2.$$

[8 marks]

7.(i) By considering the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the system of equations

$$x'(t) = 2x(t) - 4y(t); \quad y'(t) = x(t) - 3y(t).$$

Hence sketch the phase plane for the system. **[11 marks]**

(ii) Show that all solutions of the system

$$x' = ax + by \quad y' = cx + dy$$

where a, b, c and d are constants approach zero as $t \to \infty$ if $a + d < 0$ and $ad - bc > 0$. **[8 marks]**

8. Find all equilibrium points for the system

$$x' = x(1 - y); \quad y' = y(3 - x)$$

and determine the nature of each.

Hence sketch the phase plane for the system in the first quadrant.

[10 marks]

9.(i) By considering a Lyapunov function of the form $V(x, y) = ax^2 + by^2$ for an appropriate choice of a and b, show that $(0, 0)$ is a stable equilibrium point for the system

$$x' = y^3 - x^3; \quad y' = -2xy^2.$$ **[6 marks]**

(ii) Show that $(0, 0)$ is an unstable equilibrium point for the system

$$x' = x^3 - y^3; \quad y' = -2xy^2.$$ **[3 marks]**

END OF PAPER