Solutions to Mathematical Techniques Exam - June 2000

1. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = [\frac{1}{2\pi}\cdot\frac{1}{3}\,x^3]_{x=-\pi}^{x=\pi} = \frac{\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$
$= \frac{1}{\pi}[\,\frac{1}{n}\,x^2\sin(nx)|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin(nx)\,dx$
$= \frac{1}{\pi}[\,0 + \frac{2}{n^2}\,x\cos(nx)|_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi}\cos(nx)\,dx$
$= \frac{4}{n^2}\cos(n\pi) = (-1)^n\frac{4}{n^2}$.
Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$ as integrand is odd.
Hence full range Fourier series is

$$x^2 = \frac{\pi^2}{3} - 4[\,\cos(x) - \frac{1}{4}\cos(2x) + \frac{1}{9}\cos(3x) - \ldots]$$

Putting $x = 0$ we obtain

$$0 = \frac{\pi^2}{3} - 4[\,1 - \frac{1}{4} + \frac{1}{9} - \ldots].$$

Hence

$$1 - \frac{1}{4} + \frac{1}{9} - \ldots = \frac{\pi^2}{12}.$$

2. (a) $-y'' = \lambda y; \quad y(0) = 0, \;\; y'(1) = 0.$
Suppose $\lambda < 0$; then we may write $\lambda = -k^2$ where $k > 0$.
Then $y'' = k^2 y$ has general solution $y = A\,\cosh(kx) + B\,\sinh(kx)$.
We have $y' = Ak\,\sinh(kx) + Bk\,\cosh(kx)$.
$y(0) = 0 \iff A = 0$. Hence
$y'(1) = 0 \iff Bk\,\cosh(k) = 0$, i.e., $B = 0$.
Thus we must have $A = B = 0$.
Hence, if $\lambda < 0$, $y \equiv 0$ is the only solution, i.e., λ is not an eigenvalue.

Suppose $\lambda = 0$; then equation becomes $y'' = 0$ which has general solution $y = Ax + B$.
Now $y(0) = 0 \iff B = 0; \quad y'(1) = 0 \iff A = 0$.
Thus $y \equiv 0$ is the only solution and so $\lambda = 0$ is not an eigenvalue.

Suppose $\lambda > 0$; then we may write $\lambda = k^2$ where $k > 0$.

Then $y'' = -k^2 y$ has general solution $y = A\cos(kx) + B\sin(kx)$.
$y(0) = 0 \iff A = 0$. Hence
$y'(1) = 0 \iff kB\cos(k) = 0 \iff B = 0$ or $\cos(k) = 0$
$\iff B = 0$ or $k = \frac{\pi}{2} + n\pi = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2 \ldots$
Therefore $y = \sin[(n + \frac{1}{2})\pi x]$ is a nonzero solution corresponding to
$\lambda = (n + \frac{1}{2})^2\pi^2$.
Hence eigenvalues are $\frac{\pi^2}{4}$, $\frac{9\pi^2}{4}$, $\frac{25\pi^2}{4}, \ldots, (n+\frac{1}{2})^2\pi^2, \ldots$ corresponding to the

eigenfunctions $\sin(\frac{\pi x}{2})$, $\sin(\frac{3\pi x}{2})$, $\sin(\frac{5\pi x}{2}), \ldots, \sin[(n + \frac{1}{2})\pi x], \ldots$

2(b) $u_t = u_{xx}\,;\qquad u(0, t) = 0 = u_x(1, t),\quad u(x, 0) = \sin(\frac{3\pi x}{2})$.
We seek solutions of the form $u(x, t) = X(x)T(t)$.
To ensure that $u(0, t) = 0 = u_x(1, t)$ we require that $X(0) = 0 = X'(1)$.
Then u satisfies the equation $u_t = u_{xx}$ if $T'X = X''T$ and so if
$\frac{X''}{X} = \frac{T'}{T} = -\lambda$ where λ is a constant,
Thus we require

$$-X'' = \lambda X, \quad X(0) = 0 = X'(1) \qquad (1)$$
$$T' = -\lambda T. \qquad (2)$$

(1) has nonzero solutions if and only if $\lambda = \frac{\pi^2}{4}, \frac{9\pi^2}{4}, \frac{25\pi^2}{4}, \ldots$ and the corresponding solutions are $\sin(\frac{\pi x}{2}), \sin(\frac{3\pi x}{2}), \sin(\frac{5\pi x}{2}), \ldots$
Also (2) has general solution $T(t) = Ae^{-\lambda t}$.
Thus we have the non-zero solution

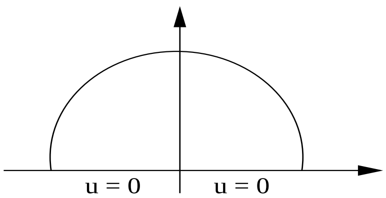
$$u(x, t) = \Sigma_{n=0}^{\infty} A_n \sin[(n + \frac{1}{2})\pi x]\, e^{-[(n+\frac{1}{2})\pi]^2 t}$$

for any choice of coefficients A_n.
Since $u(x, 0) = \Sigma_{n=0}^{\infty} A_n \sin[(n+\frac{1}{2})\pi x] = \sin(\frac{3\pi x}{2})$, it suffices to choose $A_1 = 1$ and all the other A_n's $= 0$.
Thus the required solution is $u(x, t) = \sin(\frac{3\pi x}{2})\, e^{-\frac{9\pi^2 t}{4}}$.

3. $u_x + 2xu_y = 1\,;\qquad u(0, y) = y$.
Characteristics are solutions of $\frac{dy}{dx} = 2x$, i.e., $y = x^2 + c$.
Let $(x_0, y_0) \in \mathbf{R^2}$. The characteristic through (x_0, y_0) is $y = x^2 + (y_0 - x_0^2)$ and this characteristic meets the line $x = 0$ where $y = y_0 - x_0^2$, i.e., at the point $(0, y_0 - x_0^2)$.

Let $v(x) = u(x, y(x))$ where y is the above characteristic.
Then $v(0) = u(0, y_0 - x_0^2) = y_0 - x_0^2$.
Also $\frac{dv}{dx} = u_x + u_y \frac{dy}{dx} = u_x + 2xu_y = 1$.
Hence $v(x) = x + c$ for some constant c.
Since $v(0) = y_0 - x_0^2, \quad c = y_0 - x_0^2$ and so $v(x) = x + y_0 - x_0^2$.
Hence $u(x_0, y_0) = v(x_0) = x_0 + y_0 - x_0^2$.
Thus $u(x, y) = x + y - x^2$.

4.



We seek solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$.
We require $\Theta(0) = 0$ and $\Theta(\pi) = 0$.
Also we have a solution to the equation if

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and so

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = k \text{ where } k \text{ is a constant.}$$

Thus we require

$$r^2R'' + rR' - kR = 0 \qquad (1)$$

$$-\Theta'' = k\Theta; \quad \Theta(0) = 0, \quad \Theta(\pi) = 0 \qquad (2)$$

(2) has non-zero solutions if and only if $k = n^2$ for $n = 1, 2, \ldots$ with corresponding eigenfunctions $\sin(n\theta)$.
When $k = n^2$, (1) becomes

$$r^2R'' + rR' - n^2R = 0 \quad \text{- an Euler equation}$$

which has solutions r^n and r^{-n}.
Since we require solutions to be bounded at $r = 0$, we do not make use of the solutions r^{-n}.
Thus equation has solutions $r^n \sin(n\theta)$ for $n = 1, 2, \ldots$.

Hence any function of the form

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$$

is also a solution.

Thus we require that $10 = u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin(n\theta)$ for $0 \le \theta \le \pi$ and so we choose the coefficients A_n to be the appropriate Fourier sine coefficients, i.e,

$A_n = \frac{20}{\pi}\int_0^{\pi} \sin(n\theta)\,d\theta = \frac{20}{\pi}\,.\,(-\frac{1}{n}\,\cos(n\theta)|_{\theta=0}^{\theta=\pi} = \frac{20}{n\pi}[1-\cos(n\pi)] = \begin{cases} \frac{40}{n\pi} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even.} \end{cases}$.

5. $u_t = u_{xx}$ for $-\infty < x < \infty$, $t > 0$.

Taking Fourier transforms with respect to x, we obtain

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)e^{-ix\xi}\,dx = -\xi^2\hat{u}(\xi,t)$,

i.e., $\frac{d}{dt}\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t)$ and so $\hat{u}(\xi,t) = K(\xi)e^{-\xi^2 t}$.

Now $\hat{u}(\xi,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ix\xi}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}e^{-ix\xi}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}\cos(x\xi)\,dx = \frac{2\sqrt{\pi}}{\sqrt{2\pi}}e^{-\xi^2} = \sqrt{2}e^{-\xi^2}$.

Hence $K(\xi) = \sqrt{2}e^{-\xi^2}$ and so $\hat{u}(\xi,t) = \sqrt{2}e^{-\xi^2(t+1)}$.

Thus

$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(\xi,t)e^{ix\xi}\,d\xi = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2(t+1)}e^{ix\xi}\,d\xi$

$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\xi^2(t+1)}\cos(x\xi)\,d\xi = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{t+1}}\,e^{-\frac{x^2}{4(t+1)}} = \frac{1}{\sqrt{t+1}}\,e^{-\frac{x^2}{4(t+1)}}$.

6. (i) $x' = 5x - 3y$, $y' = 6x - 4y$; $x(0) = 0$, $y(0) = 1$.

Taking Laplace transforms we obtain

$s\overline{x}(s) - x(0) = 5\overline{x}(s) - 3\overline{y}(s)$; i.e., $(s-5)\,\overline{x}(s) + 3\overline{y}(s) = 0$ (1)

$s\overline{y}(s) - y(0) = 6\overline{x}(s) - 4\overline{y}(s)$; i.e., $-6\overline{x}(s) + (s+4)\,\overline{y}(s) = 1$ (2)

$(s+4)\times$ (1) $-3\times$(2) gives $[18 + (s-5)(s+4)]\,\overline{x}(s) = -3$,

i.e., $(s^2 - s - 2)\,\overline{x}(s) = -3$, i.e., $\overline{x}(s) = \frac{-3}{(s-2)(s+1)} = -\frac{1}{s-2} + \frac{1}{s+1}$.

Hence $x(t) = -e^{-2t} + e^{-t}$.

$6 \times$ (1) $+(s-5)\times$ (2) gives $[18 + (s-5)(s+4)]\,\overline{y}(s) = s - 5$,

i.e., $(s^2 - s - 2)\,\overline{y}(s) = s - 5$, i.e., $\overline{y}(s) = \frac{s-5}{(s-2)(s+1)}$.

If $\frac{s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$, then $A(s+1) + B(s-2) = s - 5$.

Then $s=-1 \Longrightarrow -3B=-6$, i.e., $B=2$.
Also $s=2 \Longrightarrow 3A=-3$, i.e., $A=-1$.
Thus $\overline{y}(s) = -\frac{1}{s-2} + \frac{2}{s+1}$.
Hence $y(t) = -e^{2t} + 2e^{-t}$.

6. (ii) $x' = 5x - 3y, \quad y' = 6x - 4y.$

The system can be written as $\underline{x}' = A\underline{x}$ where $A = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

λ is an eigenvalue of A if and only if $\det \begin{bmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{bmatrix} = 0$,

i.e., $(5-\lambda)(-4-\lambda)+18=0$, i.e., $\lambda^2-\lambda-2=0$, i.e., $(\lambda-2)(\lambda+1)=0$, i.e., $\lambda = 2,\ -1$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$ if and only if

$$\begin{aligned} 5x - 3y &= 2x \qquad \text{i.e., } 3x - 3y = 0. \\ 6x - 4y &= 2y \qquad \text{i.e., } 6x - 6y = 0. \end{aligned}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.

$\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$ if and only if

$$\begin{aligned} 5x - 3y &= -x \qquad \text{i.e., } 6x - 3y = 0 \\ 6x - 4y &= -y \qquad \text{i.e., } 6x - 3y = 0. \end{aligned}$$

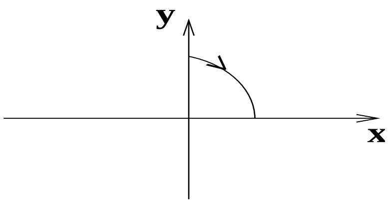
Thus $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$.

Thus system has general solution $\underline{x}(t) = c_1\, e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

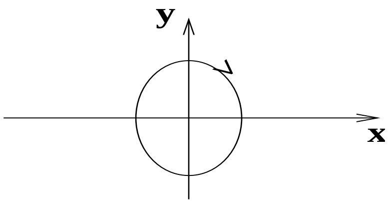
7. $x'' = -x^3$.

The equation may be written as the system $x' = y\,;\ y' = -x^3$.
Thus any trajectory of the form $y = y(x)$ must satisfy $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{-x^3}{y}$, i.e., $y\,dy = -x^3\,dx$, i.e., $y^2 + \frac{1}{2}x^4 = c$.

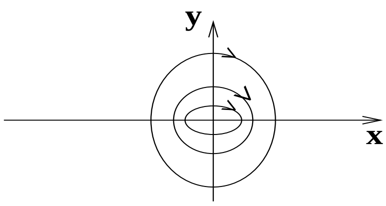
Clearly $(0,0)$ is the only equilibrium point of the system.
Consider the trajectory passing through $(0,a)$, i.e., $y^2 + \frac{1}{2}x^4 = a^2$.
As x increases into the first quadrant y decreases until $y=0$ when $\frac{1}{2}x^4 = a^2$, i.e., $x = (2a^2)^{\frac{1}{4}}$. Thus we have the trajectory



and since the equation of the trajectory is symmetric in x and y, reflecting in the x and y axes we obtain



Thus we have phase plane



8. $\frac{dx}{dt} = x(4 - x - 2y)$; $\frac{dy}{dt} = y(7 - 3x - y)$.
(x, y) is an equilibrium point iff

$$\begin{aligned} x(4 - x - 2y) &= 0 \\ y(7 - 3x - y) &= 0 \end{aligned} .$$

Hence equilibrium points occur at $(0, 0)$, $(0, 7)$, $(4, 0)$ and (x, y) where

$$\begin{aligned} x + 2y &= 4 \\ 3x + y &= 7 \end{aligned} ,$$

i.e., where $(x, y) = (2, 1)$.
If $f(x, y) = x(4 - x - 2y)$, $\frac{\partial f}{\partial x}(x, y) = 4 - 2x - 2y$ and $\frac{\partial f}{\partial y}(x, y) = -2x$.
If $g(x, y) = y(7 - 3x - y)$, $\frac{\partial g}{\partial x}(x, y) = -3y$ and $\frac{\partial g}{\partial y}(x, y) = 7 - 3x - 2y$.

Hence linearized equation at $(0,0)$ is $\underline{x}' = \begin{pmatrix} 4 & 0 \\ 0 & 7 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A are $\lambda = 4,\ 7$ and so $(0,0)$ is an unstable node.

Linearized equation at $(0,7)$ is $\underline{x}' = \begin{pmatrix} -10 & 0 \\ -21 & -7 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A are $\lambda = -10,\ -7$ and so $(0,7)$ is an asymptotically stable node.

Linearized equation at $(4,0)$ is $\underline{x}' = \begin{pmatrix} -4 & -8 \\ 0 & -5 \end{pmatrix} \underline{x} = A\underline{x}$.

Eigenvalues of A are $\lambda = -4,\ -5$ and so $(4,0)$ is an asymptotically stable node.

Linearized equation at $(2,1)$ is $\underline{x}' = \begin{pmatrix} -2 & -4 \\ -3 & -1 \end{pmatrix} \underline{x} = A\underline{x}$.

λ is an eigenvalue of A iff $(-2-\lambda)(-1-\lambda) - 12 = 0$, i.e., $\lambda^2 + 3\lambda - 10 = 0$, i.e., $(\lambda+5)(\lambda-2) = 0$, i.e., $\lambda = -5, 2$.

Hence $(2,1)$ is a saddle point.

Thus phase plane is