1. Find the full-range Fourier series for the function $f(x) = x^2$ valid for $-\pi < x < \pi$.

Hence find the sum to infinity of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$ **[7 marks]**

2. (i) Find all the eigenvalues and all the eigenfunctions of the boundary value problem

$$-y'' = \lambda y; \quad y(0) = 0, \ \ y'(1) = 0.$$ **[10 marks]**

(ii) Hence, by using the method of separation of variables, solve the heat equation

$$u_t = u_{xx}\,; \qquad 0 < x < 1, \quad t > 0$$

subject to the boundary conditions

$$u(0,t) = u_x(1,t) = 0 \text{ for all } t > 0$$

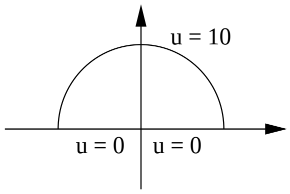
and the initial condition $u(x,0) = \sin(\frac{3\pi x}{2})$. **[10 marks]**

3. Solve the following PDE

$$u_x + 2xu_y = 1; \quad u(0,y) = y.$$ **[8 marks]**

continued overleaf

4. The steady state temperature $u(r, \theta)$ of a plate in the shape of the semi-circle $\{(r, \theta) : 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi\}$



satisfies the equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

where (r, θ) denotes plane polar co-ordinates.

If the steady state solution satisfies the boundary conditions that $u = 0$ on the x - axis and $u(1, \theta) = 10$ for $0 < \theta < \pi$, find $u(r, \theta)$. **[15 marks]**

5. By using the method of Fourier transforms, show that the equation

$u_t = u_{xx}$ for $-\infty < x < \infty$ and $t > 0$; $\quad u(x, 0) = e^{-x^2/4}$ for $-\infty < x < \infty$

has the solution $u(x, t) = \dfrac{1}{\sqrt{t+1}}\, e^{-\frac{x^2}{4(t+1)}}$. **[8 marks]**

6.(i) By taking Laplace transforms solve the initial value problem

$$x'(t) = 5x(t) - 3y(t); \quad y'(t) = 6x(t) - 4y(t); \quad x(0) = 0; \quad y(0) = 1.$$

[8 marks]

(ii) By considering the eigenvalues and eigenvectors of an appropriate matrix, find the general solution of the system of equations

$$x'(t) = 5x(t) - 3y(t); \quad y'(t) = 6x(t) - 4y(t).$$

[9 marks]

continued overleaf

7. The motion of a certain mass-spring system is governed by the equation

$$x'' = -x^3$$

where x represents the displacement from equilibrium.

Write the above equation as a system of first order equations, find all the equilibrium points and sketch the phase plane for the system.

[**12 marks**]

8. For the system describing the interaction of two competing species

$$x' = x(4 - x - 2y); \quad y' = y(7 - 3x - y)$$

find all of the equilibrium points and determine the nature of each.

Hence sketch the phase plane for the system. [**13 marks**]

END OF PAPER