

The “game of everything”

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Abstract

A type of reversible cellular automaton is proposed, which can be used to discuss the question of irreversibility in statistical mechanics. It lives on an infinite lattice in space-time, whose nodes are points with integer coordinates, either all even or all odd, and whose edges join all pairs of nearest-neighbour nodes. The states of the edges in the immediate future of each node are related to those in its immediate past by a ‘dynamical rule’ which is deterministic and invertible. The assumed probability measure makes the nodes at time 0 independent, and correlations develop as the time variable t increases because of the dynamical rule. Two types of entropy are defined: one, analogous to the Gibbs entropy in statistical mechanics, is shown to be independent of t , while the other, analogous to Boltzmann entropy in statistical mechanics, can change with t and is shown to increase in the sense that it is minimal at $t = 0$.

Key words: entropy, irreversibility, direction of time, reversible cellular automata, probability measures on infinite graphs, Gibbs states.

1 Introduction

Ever since Boltzmann’s time, the problem of irreversibility has been one of the most fundamental issues in statistical mechanics, and it is still the subject

of much discussion : see, for example, [12]. The question is this: given that the laws of mechanics, such as Newton's laws of motion, are symmetrical under time reversal, how can it be that the behaviour of real physical objects, which is supposed to be explained by these laws, does not have this symmetry? Or, in Boltzmann's words [3], "is the apparent irreversibility of all known natural processes consistent with the idea that all natural events are possible without restriction?" Boltzmann's explanation of the time-reversal asymmetry of natural processes is that "this one-sidedness lies uniquely and solely in the initial conditions", by which he means "not ... that for each experiment one must specially assume just certain initial conditions" but rather that "it is sufficient to have a uniform basic assumption about the initial properties of the mechanical picture of the world". He proposes to "conceive of the world as an enormously large mechanical system ... which starts from a completely ordered initial state, and even at present is still in a substantially ordered state".

According to this programme, then, the way to understand irreversibility is to deduce the behaviour of everything in a unified way from the reversible laws of mechanics plus a suitable assumption about the initial state of the world. This paper is part of an attempt to carry out the programme implied by these remarks of Boltzmann, using a greatly simplified model of the world. An important feature of the theory is its assumption about the initial state. This assumption is that at the initial time there are (in a precise sense) no spatial correlations. Whatever spatial correlations appear later on are produced by the action of the laws of mechanics. These laws are modelled here by treating space-time as an infinite lattice, in which the states of the edges meeting at each node are related by a local rule in the manner of cellular automata. The local rule is both deterministic and invertible with respect to time, so that the (microscopic) state of the world at any time is uniquely determined by its state at any other time, before or after. The lattice is constructed in such a way that there is an upper bound to the speed with which causal influences can travel, analogous to the speed of light in the real world; there is no instantaneous action at a distance.

The assumption that there are no spatial correlations at the initial time may seem to contradict Boltzmann's conjecture that the world 'starts from a completely ordered initial state', making it start instead from one that seems as disordered as could be. However we shall see at the end of the paper that the Boltzmann entropy of this initial state is at least as small as at any

later time, so that according to Boltzmann’s own measure of (dis)order, the entropy, this initial state is at least as ordered as at any later time. Moreover, it accords with our usual way of thinking to assume that at the beginning of time there were no spatial correlations, and that whatever correlations exist now were brought about by subsequent interactions. Our star maps are correlated with the actual positions of the stars because we have interacted with those stars by receiving light from them. But we cannot sensibly make a star map of a part of the universe so far away that the light from it has not yet reached us: until the light reaches us, the correlation that would be implied by a realistic star map does not exist.

I have called this theory “the game of everything” (GOE) because its local rule is like the rule of a board game and its automaton-like character makes it like Conway’s “Game of Life” [4], while its aim to provide a simplified model of some important features of the world as a whole could be compared with the optimistic description “a theory of everything” sometimes used in describing the search for a quantum theory of gravitation.

Among the questions one might hope to answer using the GOE theory is this: how is it that we can reliably calculate the macroscopic behaviour of an isolated system (such as a gas in a box) from its initial macro-state, giving equal probabilities to all the microstates compatible with this initial macro-state, regardless of how this initial macro-state was reached? Apparently the universe has some kind of Markovian character, so that the macroscopic behaviour of an isolated system depends (in a probabilistic sense) only on its initial macro-state and not on what happened to it before it got into that macro-state. Indeed, it is possible to take this Markovian character as one of the basic axioms of statistical mechanics [16]. But such an axiom is open to the same objection as the *Stosszahlansatz* used by Boltzmann to derive his kinetic equation for a gas: since it refers to probabilities at many different times, it may not be compatible with the equations of motion. The hope (unfortunately not realized here) of deriving a Markovian macroscopic description from something more fundamental is analogous to what Lanford actually achieved in his derivation of Boltzmann’s kinetic equation [10] for a gas using, in place of the *Stosszahlansatz*, a probability axiom which refers only to the initial time.

Within the GOE framework, there are many different possible dynamical rules, and therefore many different possible models. To show their variety, a few of the simplest ones are described in section 2. But the main concern of

this paper is to prove some general results which apply to any GOE model. The most important ones concern the entropy. Two different types of entropy are considered. One of them, analogous to the Gibbs entropy in the statistical mechanics of isolated systems, is shown to be independent of time. The other, analogous to the Boltzmann entropy, need not be constant and is shown to have a tendency to increase with time. (For the distinction between Boltzmann's definition of entropy and the statistical definition due to Gibbs, see [16]).

2 Definition of the model

The model lives on an infinite space-time graph Γ consisting of *nodes* and *edges* in R^{d+1} , where d is the number of space dimensions and the remaining dimension represents time. The nodes of Γ are points in this space-time whose $d + 1$ coordinates have integral values, which must either be all even or all odd. The edges of Γ are the line segments joining each node to all its nearest neighbour nodes, i.e. the ones whose space and time coordinates differ from its own by ± 1 .

For example, in one space dimension the nodes are the points with integer coordinates (x, t) which are such that $x + t$ is even. The edges are line segments joining nearest-neighbour nodes, i.e. from (x, t) to $(x \pm 1, t \pm 1)$ (see Figs 1 to 4).

Formally, the model includes both positive and negative values for all the space coordinates and also for the time coordinate, but we shall not use the part of Γ with negative t . The physical reason for this is that $t = 0$ represents the time when the world began. The model does yield information about the properties of the part of Γ with negative t , but they are no more than a mirror image of the corresponding properties of the part with positive t .

The second ingredient of the model is a set Ω_E of possible states for each edge of Γ . The set of conceivable states for Γ is then $(\Omega_E)^{\Gamma_E}$, where Γ_E means the set of edges in Γ . The simplest non-trivial case is for Ω_E to have two elements, e.g. {empty, occupied} in which case the state of Γ can be specified by saying which edges are occupied and which are empty. There are many other possibilities, but for simplicity it will be assumed here that the set Ω_E is finite.

Given any node with time coordinate t we define its *inputs* to be the

states of those edges connecting it to nodes with time coordinate $t - 1$, and its *outputs* to be the states of the edges connecting it to nodes with time coordinate $t + 1$. The third ingredient of the model is a *dynamical rule* relating the inputs and outputs of each node. This rule is required to be the same at each node, and to be *deterministic* (meaning that the two inputs together determine the two outputs) and *invertible* (meaning that the two outputs together determine the two inputs). The set of states of Γ that are consistent with the dynamical rule is a subset of $(\Omega_E)^{\Gamma_E}$ which we shall call Ω .

As an example, the rule could be that each output from a node is the same as the input to that node in the same space-time direction. If we think of the occupied edges as the space-time trajectories of a set of particles, the particles controlled by this rule behave like photons: they always travel at the same speed and never interact. The one-dimensional case is illustrated in Fig 1. The name ‘photon’ for this GOE system is not quite as fanciful as it might appear at first sight: in one dimension, if the state space of an edge is taken to be \mathfrak{R} instead of having only two elements, and if we denote the states of the edges with slope 1 by u and those of edges with slope -1 by v , then the variable $u + v$, evaluated at the nodes, satisfies a finite-difference approximation to the wave equation.

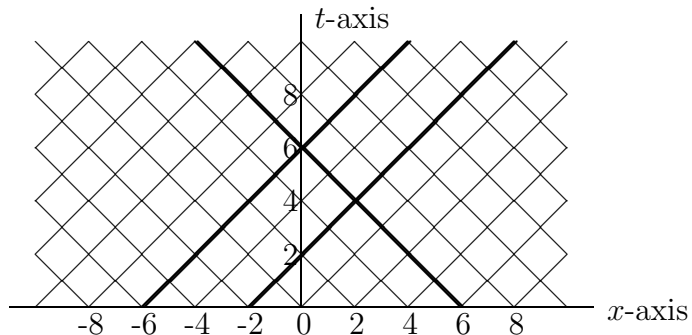


Fig 1. The ‘photon’ model in one space dimension. The occupied edges are shown thicker than the unoccupied ones.

Another possible rule is that each output is the same as the input on the same (spatial) side of the node. This time, if we think of the occupied edges as space-time trajectories of particles, each particle is trapped on a particular bond of the space lattice obtained by projecting Λ on the hyperplane $\{t = 0\}$.

The one-dimensional case (for which the projection on $\{t = 0\}$ is just the x -axis) is illustrated in Fig 2.

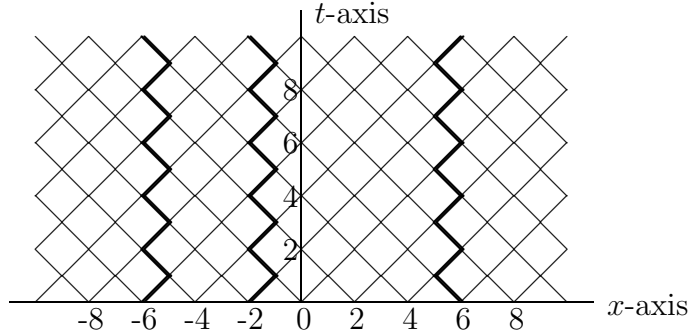


Fig 2. The ‘trapped particle’ model in one space dimension, with the same initial conditions as in Fig 1.

Fig 3 illustrates a one-dimensional variant of the ‘photon’ model in which the right-hand output is always the same as the left-hand input, but the left hand output is the same as the right-hand input if and only if the left-hand input bond is occupied. The effect of this rule, with the same initial conditions as before, is shown in Fig3. If the space dimension is taken to be periodic instead of infinite, this model is equivalent to the Kac ring model [9], [17], in which particles of two colours move in step from site to site on a ring containing fixed ‘scatterers’, and every time a particle passes a scatterer it changes colour. In Fig 3, the thick lines with slope $+1$ are the trajectories of the scatterers (which are now moving to the right rather than being fixed), the thick lines with slope -1 are the trajectories of particles of one colour and the thin lines with slope -1 are the trajectories of particles of the other colour.

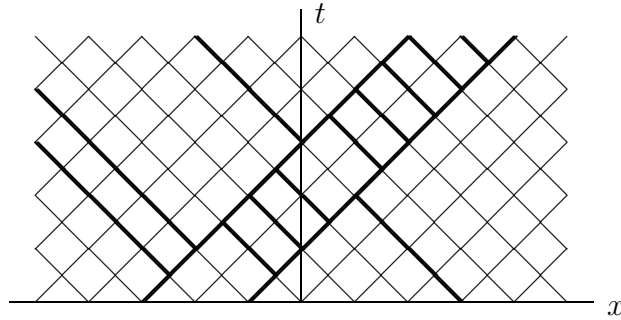


Fig 3. A GOE system analogous to the Kac ring model, with the same initial conditions as in Fig 1.

The last of the four essentially different possibilities with one space dimension and with just two states per bond is a modification of the trapped particle system in which the left-hand input and output are always the same, but the right-hand input and output are the same if and only if the left-hand input and output are occupied. For an an initial state with just one occupied bond, this rule produces a Sierpinski gasket pattern, as illustrated in Fig 4. If more than one bond is initially occupied, the patterns produced by this rule can become quite complicated.

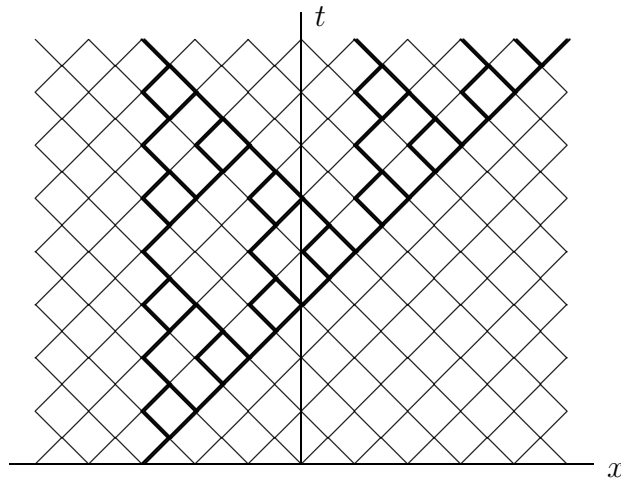


Fig 4 The ‘Sierpinski’ GOE model with just one bond occupied initially.

The GOE system need not be one-dimensional. A simple two-dimensional example is provided by the dynamic lattice gas of Hardy, de Pazzis and Pomeau [7], [8]. Here the rule is as in the ‘photon’ model described earlier

in this section, but for one change: if the inputs to a node correspond to two particles coming in from opposite directions in space, then the outputs correspond to two particles going out in the two directions at right angles to the inputs. Closely related to this ‘HPP’ model is the ‘BBM’ (billiard-ball model) reversible cellular automaton of Margolus [13], which differs from the HPP model in that if two incoming particles come in at right angles to each other, or if there are more than two incoming particles, then they go out in the same directions as they came in. Margolus shows that his model is (like Conway’s Game of Life [2]) rich enough to perform any kind of digital computation if it is started in the right initial state.

This equivalence between the Margolus cellular automaton and a GOE system is by no means accidental; it turns out that every GOE system is equivalent to a cellular automaton. The equivalence can be set up in various ways, of which the following seems the most straightforward. Define the state of a node to comprise all its inputs — or, equivalently, all its outputs. In the case of one space dimension, the state of the node at position x at time $t + 2$ is determined by the states of the nodes at positions $x - 1$ and $x + 1$ at time $t + 1$, and hence by the states of the nodes at positions $x - 2$, x and $x + 2$ at time t ; thus the model is equivalent to a certain cellular automaton whose clock ticks at even integer times and whose sites are the space points with even integer coordinates. In the case of d space dimensions, a similar construction shows that the model is equivalent to a certain d -dimensional cellular automaton whose sites are the space points all of whose coordinates are even integers.

3 Probabilities and ergodic properties

The ‘actual’ state of the model of our Universe provided by a GOE system is a particular element ω of the set Ω comprising all the states of Γ compatible with the dynamical rule. Like the state of the real Universe, some parts of which are completely empty while others contain galaxies, planets, and even living matter, this state may contain complicated structures and vary strongly with position. We shall, however, assume that if looked at on a sufficiently large scale the statistical features of the state (such as the mean density of matter in the real Universe) are independent of position. More precisely, we shall assume that there is some translationally invariant

probability measure μ such that any event having probability 1 under this measure will happen in the actual state — the actual state is “typical” with respect to this measure.

Let us define Λ to mean the set of all nodes of Γ , and Λ_t to mean the time slice consisting of all nodes whose time coordinate has the value t . For each node a , we denote by $\omega(a)$ its state, defined as before to comprise all its inputs, or equivalently all its outputs, taken together. Since the number of inputs to a node in d space dimensions is $2d$, the state space of a single node a is $\Omega_a = (\Omega_E)^{2d}$. Likewise, if A is subset of Λ , i.e. any set of nodes of Γ , we define its state $\omega(A)$ to mean the states of all the nodes in A taken together, i.e. $\omega(A) = \otimes_{a \in A} \omega(a)$. The state space of A is $\Omega_A = \otimes_{a \in A} \Omega_a$.

Because of the dynamical rule, the state of any node (x, t) is fully determined by the states of a finite set of nodes in the initial time slice Λ_0 . This finite set of nodes, which we shall call the *domain of dependence* of (x, t) and denote by $D_0(x, t)$, is

$$D_0(x, t) = \{(x', 0) \in \Lambda_0 : \|x' - x\| \leq |t|\} \quad (1)$$

where $\|x - x'\|$ means $\sum_{i=1}^d |x_i - x'_i|$ with x_i the i th coordinate of x . The notation allows for the mathematical possibility of t being negative, but this possibility is not important physically.

A measure on Ω can be specified by giving, for each finite subset A of Λ and each $\alpha \in \Omega_A$, the probabilities $\mu(\{\alpha\})$ where $\{\alpha\}$ means $\{\omega : \omega(A) = \alpha\}$. However, because of the dynamical rule, the state of each A is completely determined by the state of its domain of dependence, the finite subset of Λ_0 obtained by taking the union of the domains of dependence of the elements of A . So, to specify μ it is sufficient to specify the probabilities $\mu(\{\alpha\})$ as A runs through all the finite subsets of Λ_0 .

We take these probabilities to be those for a product measure on Λ_0 , defined by

$$\mu(\{\alpha\}) = \prod_{a \in A} p(\alpha(a)) \quad \text{for all } \alpha \in \Omega_A \quad \text{and all } A \subset \Lambda_0 \quad (2)$$

where $\alpha(a)$ means the state of node a when the set A of nodes is in state α , and $p(\cdot)$ is some probability distribution on the state space Ω_a of a single node (that is, a set of non-negative numbers $\{p(\phi) : \phi \in \Omega_a\}$ which sum to 1). The probabilities for events depending on the states of other finite subsets of Λ can then determined using the dynamical rule.

Under the probability measure μ defined in this way, the correlations at any time t have finite range. That is to say, if Λ_t denotes the set of all nodes of Λ having time co-ordinate t , then two finite subsets A and B of Λ_t are uncorrelated (statistically independent) if their spatial separation is sufficiently large :

$$\mu(\{\alpha\} \cup \{\beta\}) = \mu(\{\alpha\}) \mu(\{\beta\}) \quad \text{if } d(A, B) > 2t \quad (3)$$

where $d(A, B)$ is the spatial distance between A and B , defined by

$$d(A, B) = \min_{(x,t) \in A, (y,t) \in B} \|x - y\| \quad (4)$$

The reason is that if $d(A, B) > 2t$ the domains of dependence of the two sets A and B are disjoint, and are therefore uncorrelated under the product measure (2).

For each time t , let us denote by Ω_t the state space of the time slice Λ_t , that is, $\Omega_t = \otimes_{a \in \Lambda_t} \Omega_a$. The probability measure on Ω_0 defined by (2) can be thought of as a Gibbs measure corresponding to the formal Hamiltonian

$$H = \sum_{a \in \Lambda_0} \Phi_0(\omega(a)), \quad (5)$$

where

$$\Phi_0(\alpha) = -\log p(\alpha) \quad (\alpha \in \Omega_a) \quad (6)$$

It turns out that the measure on the state space Ω_t of a general time slice Λ_t is also a Gibbs measure. Since the dynamical rule is deterministic and reversible, each state of Λ_t corresponds to exactly one state of Λ_0 . To find the state at time 0 from which a given state at time t came, we can apply the dynamical rule in reverse to find the state of each node in Λ_0 separately. The state of the node $(x, 0)$ is a function of the state of its ‘inverse domain of dependence’ at time t , that is to say a function of the state of the set of nodes

$$D_t(x, 0) = \{(x', t) \in \Lambda_t : \|x' - x\| \leq |t|\} \quad (7)$$

Denoting this function by $g_t(\cdot)$, we have

$$\omega(x, 0) = g_t(\omega(D_t(x, 0))) \quad \text{for all } \omega \in \Omega \quad (8)$$

So the formal Hamiltonian given in (5) can be rewritten in terms of the state of Λ_t :

$$H = - \sum_{a \in \Lambda_0} \Phi_t(\omega(D_t(a))) \quad (9)$$

where $\Phi_t(\cdot)$ is defined by

$$\Phi_t(\alpha) = \Phi_0(g_t(\alpha)) \quad (\alpha \in \Omega_{D_t(x,0)}) \quad (10)$$

Since the dynamical rule implies a one-to-one correspondence between the state at time 0 and the state at time t , the probability distribution at time t is formally $\exp(-H)$ with H given by (9). Each term in the right side of (9) depends only on the state of a set of vertices in Λ_t , namely the set $D_t(a)$, $a \in \Lambda_0$, which has diameter $2t$ in the $\|\cdot\|$ norm. In other words, the interactions in this Hamiltonian have the finite range $2t$, and so the measure on Λ_t is a Gibbs measure, obeying the DLR condition [11].

The product measures on Λ_0 defined by (2) are invariant under the spatial translation operators X_1, \dots, X_d defined by the relation

$$X_i \mu(\{\omega : \omega(A) = \alpha\}) = \mu(\{\omega : \omega(A + 2\mathbf{e}_i) = \alpha\}) \quad (\alpha \in \Omega_A) \quad (11)$$

where $A + 2\mathbf{e}_i$ is the set of nodes obtained by shifting each element of A by a distance of two units in the direction of the i th coordinate axis. Since the dynamical rule is also invariant under spatial translations, it follows that the measure on Λ obtained by combining (2) with the dynamical rule is invariant under such translations.

In addition to being invariant under space translations, the product measures are (by (3)) mixing, and therefore ergodic, with respect to each of the space translation operators X_i . It follows that the measures obtained by applying the dynamical rule to these product measures are also ergodic and mixing with respect to each of the space translations. The ergodic property that concerns us here is

$$\lim_{L \rightarrow \infty} \frac{1}{(2L)^d} \sum_{-L < n_1 \leq L} \sum_{-L < n_2 \leq L} \dots \sum_{-L < n_d \leq L} \delta(\alpha, \omega(X_1^{n_1} X_2^{n_2} \dots X_d^{n_d} A)) = \mu(\alpha) \quad \text{almost surely} \quad (12)$$

where A is any finite subset of Λ , α is any state of A (i.e. any element of Ω_A), and $\delta(\alpha, \beta)$ is defined to be 1 if $\alpha = \beta$ and 0 otherwise.

It is sometimes argued that we should not apply probability theory to the universe as a whole, since the universe is a unique object : it was not (so far as we can ever know) drawn at random from some ensemble of universes. Equation (12) shows, however, how probability theory can still be quantitatively useful even though our universe is unique. If it is found empirically that limits of the kind shown on the left hand side of (12) appear to exist, than a reasonable explanation is that the actual universe is ‘typical’ with respect to some probability distribution which is invariant and ergodic with respect to space translations. Here saying that the actual universe is ‘typical’ means that if the probability distribution implies that a certain event has probability 1, then that event does happen in the actual Universe. Moreover, eqn (12) then gives us, in principle, a method of measuring the relevant probability distribution — or, if the measurements are too difficult to carry out in practice, at least it gives us a method of defining the probability distribution in terms of empirically measurable quantities in a unique universe. This method of defining probabilities in a unique universe, using a space ensemble, was suggested in [15] but the idea goes back much further, e.g. [14].

Under the recipe (2), each single-node initial probability distribution p gives a different measure μ , but the case where $p(\cdot)$ makes all the states in Ω_a equally probable is particularly important. For this measure the nodes in every time slice, not just those in the one at time 0, are statistically independent, that is to say eqn (2) holds with Λ_0 replaced by Λ_t for any integer t . This measure will be called here the *equiprobable measure*.

4 Time translations and equilibrium measures

In addition to the space translations X_i defined in (11) we can also consider a time translation operator T defined by

$$T\mu(\{\omega : \omega(A) = \alpha\}) = \mu(\{\omega : \omega(A + 2\mathbf{e}_0) = \alpha\}) \quad (\alpha \in \Omega_A) \quad (13)$$

where $A + 2\mathbf{e}_0$ is the set of nodes obtained by shifting each element of A by a distance of two units in the direction of the time axis.

A measure that is invariant under T will be called an *equilibrium measure*. It is easily checked that the equiprobable measure is an equilibrium measure. For some dynamical rules there are other equilibrium measures as well. For

example, in the ‘photon’ model illustrated in Fig 1, there is an equilibrium measure for which the measure in the initial time slice Λ_0 (and, consequently in all other time slices as well) is of the form (2) with p given by

$$p(\alpha) = q(u)r(v). \tag{14}$$

Here $u \in \{0, 1\}$ is the input from the left and v is the input from the right, while $q(u)$ means $q_0(1 - u) + q_1u$ with q_0 and q_1 non-negative numbers adding to 1, and $r(u) = r_0(1 - v) + r_1v$ analogously. Of course q_0 is the probability of the event $u = 0$, r_1 is that of $v = 1$, and so on. Any allowed values for q_0, q_1, r_0, r_1 give an equilibrium measure; the special values $q_0 = q_1 = r_0 = r_1 = \frac{1}{2}$ give the equiprobable measure.

The ‘trapped particle’ model illustrated in Fig 2 also has equilibrium measures of this type, but only if $q_0 = r_0$ (so that $q_1 = r_1$, i.e. all bonds are equally likely to be occupied). The ‘Kac’ model illustrated in Fig 3 has such equilibrium measures too; this time q_0 is arbitrary but r_0 and r_1 must equal $\frac{1}{2}$, that is, the left-moving particles are equally likely to be of either colour. The ‘Sierpinski’ model illustrated in Fig 4 has, however, no equilibrium measures of this type other than the equiprobable measure.

There appears to be a link between these non-equiprobable product measures and conservation laws. For the photon system, there are two conserved quantities at each node — the numbers of photons travelling in each direction —, and there are two arbitrary numbers q_0 and r_0 in the formula for the product measure. For the trapped particle model there is one conserved quantity at each node, the number of particles, and there is one arbitrary number in the formula. For the ‘Kac’ model there is again one conserved quantity at each node, the number of particles moving to the right, and again one arbitrary number in the formula. For the ‘Sierpinski’ model there are no conserved quantities and no non-equiprobable equilibrium measures.

Even for the very simple systems considered in these examples, there can also be equilibrium measures which are not product measures. The ‘Kac’ model provides an example. If its initial probability distribution has the form (2) with p given by (14) with q_0 and r_0 both different from $\frac{1}{2}$, then the probability distribution on Λ_t approaches a limit as $t \rightarrow \infty$ which is not a product measure. In this measure the probabilities for the two possible inputs from the right to a given node are $\frac{1}{2}$, as in the equilibrium product measure, but it is not a product measure because the different nodes in a time slice are correlated.

To demonstrate that this correlation exists consider two neighbouring left-moving particles. The probability that they have the same colour at time 0 is $r_0^2 + r_1^2$. The probability that during the time interval from 0 to t they will both meet the same number of right-moving particles is $q_0^2 + q_1^2$, and so the probability that they still have the same colour at time t is $(r_0^2 + r_1^2)(q_0^2 + q_1^2)$. Add to this the probability that they started with different colours and that one of them met one more right-moving particle than the other, which is $(2r_0r_1)(2q_0q_1)$, and we find that the total probability for the two particles to have the same colour is $\frac{1}{2}[1 + (r_1 - r_0)^2(q_1 - q_0)^2] \neq \frac{1}{2}$, regardless of the value of t . So, however large t is, the colours of the two neighbouring left-moving particles remain correlated, even though for each of them individually the two colours eventually become equally likely.

5 Entropy

Given any finite subset A set of the GOE lattice of nodes Λ , we can define its entropy in the usual way (see, for example, [1]), as

$$S(A) := - \sum_{\alpha \in \Omega_A} \mu_A(\{\alpha\}) \log \mu(\{\alpha\}) \quad (15)$$

with the convention $0 \log 0 = 0$. Given another finite set of nodes B , the conditional entropy of A with respect to B is defined as

$$\begin{aligned} S(A|B) &= - \sum_{\alpha \in \Omega_A} \sum_{\beta \in \Omega_B} \mu(\{\alpha\} \cap \{\beta\}) \log \frac{\mu(\{\alpha\} \cap \{\beta\})}{\mu(\{\beta\})} \\ &= S(A \cup B) - S(B) \end{aligned} \quad (16)$$

where $\{\beta\}$ means $\{\omega : \omega(B) = \beta\}$. Since the logarithms are non-negative, it follows from this definition that

$$S(A \cup B) \geq S(B) \quad (17)$$

i.e. the entropy is a non-decreasing set function. Also, it can be shown using Jensen's inequality that $S(A|B) \leq S(A)$, and hence by (16) that S is a sub-additive set function:

$$S(A \cup B) \leq S(A) + S(B) \quad (18)$$

Let A and B be any two subsets of Λ such that the state of A is determined by the state of B . We shall write this relationship $A \preceq B$. The relationship holds whenever $A \subset B$, but also in other cases, in particular when B is the domain of dependence of A as defined in section 3 above. Whenever $A \preceq B$ the set of dynamically admissible states for B is the same as the corresponding set for $A \cup B$ and the probabilities of these states are also the same, so that $S(A|B) = 0$ by (16). Since also $S(B|A) \geq 0$ eqn (16) gives

$$\left. \begin{aligned} S(A|B) &= 0 \\ S(B) - S(A) &= S(B|A) \geq 0 \end{aligned} \right\} \text{if } A \preceq B \quad (19)$$

If A consists only of nodes in time slice 0, then the statistical independence rule (2) implies that $S(A)$ is equal to the number of nodes in A multiplied by the entropy of a single node at time 0, which we denote by s_0 :

$$s_0 := - \sum_{\omega \in \Omega_E} p(\omega) \log p(\omega) \quad (20)$$

Consider now a set A of nodes in a general time slice Λ_t . The nodes are no longer statistically independent, but we can still (following Goldstein [5]) define an entropy per site in the limit where A is a very large hypercube:

$$s(t) = \lim_{n \rightarrow \infty} \frac{S(A_n(t))}{\#(A_n(t))} \quad (21)$$

where

$$A_n(t) = \bigcup_{-n < x_1 \leq n} \dots \bigcup_{-n < x_d \leq n} (x_1, \dots, x_d, t) \quad (22)$$

and $\#(A_n(t))$ denotes the number of nodes in $A_n(t)$, namely n^d . Since this definition reduces in the case of one space dimension to the Kolmogorov-Sinai entropy of the probability measure with respect to the space translation operator X_1 , we shall call it the Kolmogorov-Sinai entropy per site.

For the case $t = 0$ the right side of (21) is equal to s_0 as defined in eqn (20) for all n as well as in the large- n limit. The following theorem gives us the value of the K-S entropy at any other time :

Theorem 1 *The Kolmogorov-Sinai entropy per site at time t is independent of t and is equal to s_0 , i.e.*

$$s(t) = s_0 \text{ for all } t \quad (23)$$

Proof: We have already seen that the result is true for $t = 0$. For other values of t , consider any n greater than t and define $A_{n-t}(0)$ and $A_{n+t}(0)$ in accordance with eqn (22). Our dynamical rule ensures that $A_{n-t}(0) \preceq A_n(t) \preceq A_{n+t}(0)$. Since the nodes at time 0 are statistically independent, the entropies of the sets $A_{n-t}(0)$ and $A_{n+t}(0)$ are respectively $\#(A_{n-t}(0))s_0 = (n-t)^d s_0$ and $\#(A_{n+t}(0))s_0 = (n+t)^d s_0$. Hence, by (19), we have

$$(n-t)^d s_0 \leq S(A_n(t)) \leq (n+t)^d s_0 \quad (24)$$

On dividing by $\#(A_n(t)) = n^d$, taking the limit $n \rightarrow \infty$ and using (21), the result follows. Q.E.D.

Physically, the reason why the K-S entropy per site, as defined above, is independent of t is that its definition refers to very large sets of nodes. Even though some of the information contained in the initial conditions is escaping with the speed of light, the very large sets of nodes used in the definition (21) of $s(t)$ are large enough to “recapture” all the information. No information about the initial state is lost, and so the entropy, defined as above using arbitrarily large subsets of Λ , does not increase. In this respect the K-S entropy is analogous to the microscopic Gibbs entropy $\int \rho \log(1/\rho)$ of statistical mechanics. For an isolated system, the Gibbs entropy is independent of time, and the physical reason for this constancy is essentially the same as in our case : since the system is isolated, no information can escape.

In contrast to the K-S entropy, which concerns the probability distributions of arbitrarily large sets of nodes at a fixed time, the entropy of a set of nodes with fixed size does have a tendency to increase with time. This tendency made precise in the following theorem.

Theorem 2

$$S(A_n(t)) \geq S(A_n(0)) \quad (25)$$

Proof: The proof depends on putting together $A_n(t)$ and $2^d - 1$ of its spatial translates to make a set which is a spatial translate of $A_{2n}(t)$. These 2^d sets are obtained by applying to $A_n(t)$ the translation operators $(Z_r)^n$, where Z_r is defined by

$$Z_r = \prod_{i=1}^d (X_i)^{r_i} \quad (r = 0, \dots, 2^d - 1) \quad (26)$$

where r_i is the i th digit in the binary representation of the integer r . The sets $(Z_r)^n A_n(t)$ are disjoint, and their union is a space translate of $A_{2n}(t)$ and

therefore has the same entropy as $A_{2n}(t)$. It follows, by the sub-additivity property (18) of the entropy and the invariance of entropy under spatial translations, that

$$\begin{aligned} S(A_{2n}(t)) &\leq \sum_{r=0}^{2^d-1} S((Z_r)^n A_n(t)) \\ &= 2^d S(A_n(t)) \end{aligned} \quad (27)$$

and hence that

$$\frac{S(A_{2n}(t))}{\#(A_{2n}(t))} \leq \frac{S(A_n(t))}{\#(A_n(t))} \quad (28)$$

Replacing n in the above argument by $2n$, then by $4n$, and so on, and combining the results, we find that

$$\frac{S(A_{2^k n}(t))}{\#(A_{2^k n}(t))} \leq \frac{S(A_n(t))}{\#(A_n(t))} \quad (29)$$

for any positive integer k . But by the definition (21) of KS entropy, the left side of (29) approaches the limit $s(t)$ as k becomes very large. The desired result now follows by using Theorem 1, Q.E.D.

This result is easily generalized to some other shapes for the set A , for example (in two dimensions) rectangles.

Theorem 2 shows that $S(A_n(t))$ increases with t in the sense of being larger than its initial value, but not that the increase is monotonic. There are some cases where the increase is indeed monotonic, but there are others where it is not. For example, in the ‘Kac’ model of section 2 $S(A_1(t))$ increases monotonically to an equilibrium value, but in the ‘trapped particle’ model it does not go to equilibrium at all, being periodic with period 2. In the ‘Sierpinski’ model a long calculation shows that $S(A_1(t))$ approaches an equilibrium value, but not monotonically.

The tendency of $S(A_n(t))$ to increase with time suggests that this quantity may be related to Boltzmann entropy, which also tends to increase with time. Such a relationship has been found by S. Goldstein [5]. Consider a large block of nodes at time t , namely $A_n(t)$ for some large number n , and define its macro-state as the function giving the distribution of these nodes over the single-node state space Ω_a , i.e. the function f defined by

$$f(\alpha) = \#\{b \in A_n(t) : \omega(b) = \alpha\} \quad (\alpha \in \Omega_a) \quad (30)$$

The Boltzmann entropy of this macro-state is (in units where Boltzmann's constant equals 1)

$$\begin{aligned} S_B(A_n(t)) &= \log \frac{(\sum_{\alpha} f(\alpha))!}{\prod_{\alpha} f(\alpha)!} \\ &\approx -\sum_{\alpha} f(\alpha) \log \frac{f(\alpha)}{\sum_{\alpha} f(\alpha)} \end{aligned} \quad (31)$$

by Stirling's formula. In the large- n limit we obtain, using the ergodic property (12),

$$\lim_{n \rightarrow \infty} \frac{S_B(A_n(t))}{\#(A_n(t))} = S(A_1(t)) \quad \text{almost surely} \quad (32)$$

Thus the two different types of entropy in the GOE model correspond to the two different types of entropy, Gibbs and Boltzmann, that are used in non-equilibrium statistical mechanics. Theorem 1 shows that the analogue of Gibbs entropy is constant in time; Theorem 2, when combined with eqn (32), that the analogue of Boltzmann entropy tends to increase with time.

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