Bose-Einstein Condensation & Phonons in Helium II

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This paper discuusses the superfluidity of liquid helium below 0.5^o K in a periodic box. The main results are (i) that B.E. condensation is present and (ii) that Landau's method of defining $\mathbf{v}_s$, the superfluid velocity, is equivalent to London and Tisza's

It is assumed that, under the two restrictions mentioned, all important stationary states can be accurately enough constructed using the phonon theory (1). Landau has shown how to describe superfluid flow, with $\mathbf{v}_s$ identified as the velocity of a moving ground state and $\mathbf{v}_n$ as the velocity of a phonon gas built on the moving grund state. However, his theory does not explain whiy non-conservative collisions between phonons should not produce friction between the two fluids, nor does it explain the observed critical velocity effects.

For a more complete decription than Landau's, one can incorporate London & Tisza's ideas concerning Bose-Einstein condensation into the theory. In fact, it has been shown (2) that B.E. condensation is present in He II at absolute zero: one object of the present work is to extend this result to finite temperatures without using the very crude method of section 7 of ref.2. If the presence of B.E. condensation can be demonstrrated, then London & Tisza's ideas suggest that $\mathbf{v}_s$ will equal the velocity of the condensed particles. For consistency, thaerefore, one must show that this interpretation of $\mathbf{v}_s$ agrees with Landaplcu's. This was the second object of the present work.

In the phonon theory, approximate stationary states are construced by adding phonons to the ground state ψ or to 'moving gorund states' of the form

$$\psi_{\mathbf{u}} := \exp\left(im\mathbf{u}\cdot\sum_j \mathbf{q}_j/\hbar\right)\psi \qquad (1)$$

where the $\mathbf{q}_j$'s are the positioin vectors of the N atoms, and $\mathbf{u}$ is chosen to make $m\mathbf{u}$ an allowed single particle momentum. Feynman (3) has shown that the normalized state obtained from ψ by adding a single phonon in the mode $\mathbf{k}$ is approximately $F_{\mathbf{k}}\psi$, where

$$F_{\mathbf{k}} := (2mc/N\hbar k)^{1/2}\rho_{\mathbf{k}} := (2mc/N\hbar k)^{1/2}\sum_j \mathrm{e}^{i\mathbf{k}\cdot\mathbf{q}_j} \tag{2}$$

in which c denotes the speed of sound. Oue first step is to generalize this to the case where, for each $\mathbf{k}$, $n_{\mathbf{k}}$ phonons are added successively. Using the methods of ref. (2), it can be shown that the excited state has the form

$$\psi_{ex} = \left\{ \left(\prod_{\mathbf{k}}\right)^{1/2} \mathbf{e}^{F_{\mathbf{k}}F_{-\mathbf{k}}} \left(\frac{\partial}{\partial F_{-\mathbf{k}}}\right)^{n_{\mathbf{k}}} \left(\frac{\partial}{\partial F_{\mathbf{k}}}\right)^{n_{-\mathbf{k}}} \mathrm{e}^{-F_{\mathbf{k}}F_{-\mathbf{k}}} \right\} \psi \tag{3}$$

where $\left(\prod_{\mathbf{k}}\right)^{1/2}$ means a product where each pair $(\mathbf{k}, -\mathbf{k})$ is included once only. The expression in braces is simply a convenient representation for a certain polynomial in the $\rho_{\mathbf{k}}$'s, analogous to a Hermite polynomial. The momentum of the state ψ_{ex} is $\mathbf{M} := \sum_{\mathbf{k}} \hbar\mathbf{k}n_{\mathbf{k}}$; its energy is $E := E_0 + \sum_{\mathbf{k}} \hbar kcn_{\mathbf{k}}$, where E_0 is the ground-state energy; its normalization integral is $\prod_{\mathbf{k}}(n_{\mathbf{k}}!)$.

From these stationary states one may construct a density matrix, giving each state ψ_{ex} a probability proportional to the value of $\exp[-\beta(E - \mathbf{v}\cdot\mathbf{M})]$ for that state, where $\beta := 1/kT$. and $\mathbf{v}$ is a velocity which Landau identifies with $\mathbf{v}_n$. In contrast with the density matrix of a Gibbs distribution, we do not inclued all stationary states, but only those built from the true ground state (not moving ground states). the resulting distribution we call a *Landau distribution.* In general, a Landau distribution is characterized by two velocity parameters, ($\mathbf{v}_n$ and $\mathbf{v}_s$: here $\mathbf{v}_s = 0$) while the Gibbs distribution for a moving system is characterized by only one velocity.

The density matrix for a Landau distribution can now be written down; it is

$$\langle \mathbf{Q}'|\sigma|\mathbf{Q}''\rangle \propto \psi(\mathbf{Q}')\psi(\mathbf{Q}''))\times$$
$$\left(\prod_{\mathbf{k}}\right)^{1/2} \left\{ \mathrm{e}^{F'_{\mathbf{k}}F'_{-\mathbf{k}}+F''_{\mathbf{k}}F''_{-\mathbf{k}}} \exp\left[\gamma_{\mathbf{k}}\frac{\partial^2}{\partial F'_{-\mathbf{k}}\partial F''_{\mathbf{k}}} + \gamma_{-\mathbf{k}}\frac{\partial^2}{\partial F'_{\mathbf{k}}\partial F''_{-\mathbf{k}}}\right] \mathrm{e}^{-F'_{\mathbf{k}}F'_{-\mathbf{k}}-F''_{\mathbf{k}}F''_{-\mathbf{k}}} \right\} \tag{4}$$

where $\mathbf{Q}'$ means $\mathbf{q}'_1, \ldots, \mathbf{q}'_N$, $F'_{\mathbf{k}} := (2mc/\hbar k)^{1/2}\sum_j \exp(i\mathbf{k}\cdot\mathbf{q}')$, etc., and $\gamma_{\mathbf{k}} := \exp[-\beta\hbar(kc - \mathbf{k}\cdot\mathbf{v})]$. We can evaluate the product with the help of the identity

$$\mathrm{e}^{-F_{\mathbf{k}}F_{-\mathbf{k}}} = \pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}{-z\bar{z} + i(zF_{\mathbf{k}} + \bar{z}F_{-\mathbf{k}})}dxdy \tag{5}$$

where $z := x + iy, \bar{z} := x - iy$. The result is

$$\langle \mathbf{Q}'|\sigma|\mathbf{Q}''\rangle \propto \psi(\mathbf{Q}')\psi(\mathbf{Q}''))\times$$
$$\exp\left\{ -\tfrac{1}{2}\sum_i\sum_j [f(\mathbf{q}'_i - \mathbf{q}'_j) + f(\mathbf{q}''_i - \mathbf{q}''_j) + g(\mathbf{q}'_i - \mathbf{q}''_j)] \right\}, \tag{6}$$

where $f(\mathbf{r}) := (V/8\pi^3 N)\int d^3\mathbf{k}(2mc/\hbar k)(\mathrm{e}^{2\hbar\beta kc}-1)^{-1}\cos(\mathbf{k}\cdot\mathbf{r})$ and $g(\mathbf{r}) := (V/8\pi^3 N)\int d^3\mathbf{k}(2mc/\hbar k)\mathrm{cosech}(\hbar\beta kc)\cos(\mathbf{k}\cdot(\mathbf{r}-i\beta\hbar\mathbf{v}),$
V being the volume of the container. The functions $f(\mathbf{r})$ and $g(\mathbf{r})$ are both very small if $r \gg \beta\hbar c$.

To test for B.E. condensation, we construct from (6) the one-particle reduced density matrix

$$\begin{aligned}\langle\mathbf{q}'|\sigma_1|\mathbf{q}''\rangle &= N\int d\mathbf{Q}_1\langle\mathbf{q}',\mathbf{Q}_1|\sigma|\mathbf{q}'',\mathbf{Q}_1\rangle \\ &\propto \exp[\frac{1}{2}g(\mathbf{q}'-\mathbf{q}'')]\int d\mathbf{Q}_1\chi_{ex}(\mathbf{q}',\mathbf{Q}_1)\bar{\chi}_{ex}(\mathbf{q}'',\mathbf{Q}_1)\theta_{ex}^2(\mathbf{Q}_1), \quad (7)\end{aligned}$$

where $\mathbf{Q}_1$ means $\mathbf{q}_2\ldots\mathbf{q}_N$ (denoted by a German $\mathbf{q}$ in ref. (2)),
$\chi_{ex}(\mathbf{q}',\mathbf{Q}_1) := \chi(\mathbf{q}',\mathbf{Q}_1)\prod_{j=2^N}\mathrm{e}^{f(\mathbf{q}'-q_j)-(1/2)g(\mathbf{q}'-\mathbf{q}'_j)},$
$\theta_{ex}(\mathbf{Q}_1) := \theta(\mathbf{Q}_1)\exp\left\{-(1/2)\sum_{i=2}^N\sum_{j=2}^N[f(\mathbf{q}_i-\mathbf{q}_j+(1/2)g(\mathbf{q}_i-\mathbf{q}_j)]\right\},$
$\bar{\chi}_{ex}$ is the complex conjugate of χ_{ex}, and the functions χ and θ are defined in section 5 of ref. 2.

The integral in (7) can be studied by the method of section 6 of ref.2, making use of the fact that $f(\mathbf{r})$ and $g(\mathbf{r})$ are small for large r. The analysis shows that for large $|\mathbf{q}'-\mathbf{q}''|$ the integral is approximately constant, and that

$$\langle\mathbf{q}'|\sigma_1|\mathbf{q}''\rangle \approx \text{const. for large } |\mathbf{q}'-\mathbf{q}''| \quad (8)$$

where the constant is independent of N and is positive. It follows, by section 4 of ref. 2 that B.E. condensation is present and that the wave function of the condensed particles is a constant.

To show that the two methods of defining $\mathbf{v}_s$ are consistent, we consider the effect of multiplying all wave functions by $\exp(im\mathbf{u}\cdot\sum_j\mathbf{q}_j/\hbar)$. This transformation increases the momentum of every particle by $m\mathbf{u}$ and hence changes the velocity of the ground state to $\mathbf{u}$, so that Landau's definition of $\mathbf{v}_s$ now gives $\mathbf{v}_s = \mathbf{u}$. The wave function of the condensed particles is transformed to $\Psi(\mathbf{q}) = \text{const.}\exp(im\mathbf{u}\cdot\mathbf{q}/\hbar$. According to the ideas of London nd Tisza, one should define $\mathbf{v}_s$ as $(\hbar/im)\nabla(\mathbf{Im}\log\Psi(\mathbf{q}))$. Clearly this definition also gives $\mathbf{v}_s = \mathbf{u}$. Thus, under the special simplifying assumptions used here, the two definitions of $\mathbf{v}_s$ are equivalent.

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References

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