Mathematical Induction

11

11-1. INFORMAL INTRODUCTION

The point of metatheory is to establish facts about logic, as distinguished from using logic. Sentence and predicate logic themselves become the object of investigation. Of course, in studying logic, we must use logic itself. We do this by expressing and using the needed logical principles in our metalangauge. It turns out, however, that to prove all the things we want to show about logic, we need **more** than just the principles of logic. At least we need more if by 'logic' we mean the principles of sentence and predicate logic which we have studied. We will need an additional principle of reasoning in mathematics called *Mathematical Induction.*

You can get the basic idea of mathematical induction by an analogy. Suppose we have an infinite number of dominos, a first, a second, a third, and so on, all set up in a line. Furthermore, suppose that each domino has been set up close enough to the next so that if the prior domino falls over, it will knock over its successor. In other words, we know that, for all n, if the nth domino falls then the n + 1 domino will fall also. Now you know what will happen if you push over the first domino: They will all fall.

To put the idea more generally, suppose that we have an unlimited or infinite number of cases, a first case, a second, a third, and so on. Suppose that we can show that the first case has a certain property. Furthermore, suppose that we can show, for all n, that if the nth case has the property, then the n + 1 case has the property also. Mathematical induction then licenses us to conclude that all cases have the property.

If you now have the intuitive idea of induction, you are well enough prepared to read the informal sections in chapters 12 and 13. But to master the details of the proofs in what follows you will need to understand induction in more detail.

11–2. THE PRINCIPLE OF WEAK INDUCTION

Let's look at a more specific example. You may have wondered how many lines there are in a truth table with n atomic sentence letters. The answer is 2^n. But how do we prove that this answer is correct, that for all n, an n-letter truth table has 2^n lines?

If n = 1, that is, if there is just one sentence letter in a truth table, then the number of lines is $2 = 2^1$. So the generalization holds for the first case. This is called the *Basis Step* of the induction. We then need to do what is called the *Inductive Step*. We assume that the generalization holds for n. This assumption is called the *Inductive Hypothesis*. Then, using the inductive hypothesis, we show that the generalization holds for n + 1.

So let's assume (inductive hypothesis) that in an n-letter truth table there are 2^n lines. How many lines are there in a truth table obtained by adding one more letter? Suppose our new letter is 'A'. 'A' can be either true or false. The first two lines of the n + 1 letter truth table will be the first line of the n-letter table plus the specification that 'A' is true, followed by the first line of the n-letter table plus the specification that 'A' is false. The next two lines of the new table will be the second line of the old table, similarly extended with the two possible truth values of 'A'. In general, each line of the old table will give rise to two lines of the new table. So the new table has twice the lines of the old table, or $2^n \times 2 = 2^{n+1}$. This is what we needed to show in the inductive step of the argument.

We have shown that there are 2^n lines of an n-letter truth table when n = 1 (basis step). We have shown that if an n-letter table has 2^n lines, then an n + 1 letter table has 2^{n+1} lines. Our generalization is true for n = 1, and if it is true for any arbitrarily chosen n, then it is true for n + 1. The princple of mathematical induction then tells us we may conclude that it is true for all n.

We will express this principle generally with the idea of an *Inductive Property*. An inductive property is, strictly speaking, a property of integers. In an inductive argument we show that the integer 1 has the inductive property, and that for each integer n, if n has the inductive property, then the integer n + 1 has the inductive property. Induction then licenses us to conclude that all integers, n, have the inductive property. In the last

example, *All n letter truth tables have exactly 2^n lines*, a proposition about the integer n, was our inductive property. To speak generally, I will use '**P**(n)' to talk about whatever inductive property might be in question:

Principle of Weak Induction

a) Let **P**(n) be some property which can be claimed to hold for (is defined for) the integers, n = 1, 2, 3, . . . (the *Inductive Property*).
b) Suppose we have proved **P**(1) *(Basis Step)*.
c) Suppose we have proved, for any n, that if **P**(n), then **P**(n + 1) (*Inductive Step*, with the assumption of **P**(n), the *Inductive Hypothesis*).
d) Then you may conclude that **P**(n) holds for all n from 1 on.
e) If in the basis step we have proved **P**(i), we may conclude that **P**(n) holds for n = i, i + 1, i + 2,. . . .

(e) simply says that our induction can really start from any integer, as long as the inductive property is defined from that integer onward. Often it is convenient to start from 0 instead of from 1, showing that **P**(n) holds for n = 0, 1, 2,. . . .

Most of the inductions we will do involve facts about sentences. To get you started, here is a simple example. The conclusion is so obvious that, ordinarily, we would not stop explicitly to prove it. But it provides a nice illustration and, incidentally, illustrates the fact that many of the generalizations which seem obvious to us really depend on mathematical induction.

Let's prove that if the only kind of connective which occurs in a sentence logic sentence is '~', then there is a truth value assignment under which the sentence is true and another in which it is false. (For short, we'll say that the sentence "can be either true or false.") Our inductive property will be: *All sentences with n occurrences of '~' and no other connectives can be either true or false.* A standard way of expressing an important element here is to say that we will be *doing the induction on the number of connectives,* a strategy for which you will have frequent use.

We restrict attention to sentences, **X**, in which no connectives other than '~' occur. Suppose (basis case, with n = 0) that **X** has no occurrences of '~'. Then **X** is an atomic sentence letter which can be assigned either t or f. Suppose (inductive hypothesis for the inductive step) that all sentences with exactly n occurrences of '~' can be either true or false. Let **Y** be an arbitrary sentences with n + 1 occurrences of '~'. Then **Y** has the form ~**X**, where **X** has exactly n occurrences of '~'. By the inductive hypothesis, **X** can be either true or false. In these two cases, ~**X**, that is, **Y**, is, respectively, false and true. Since **Y** can be any sentence with n + 1 occurrences of '~', we have shown that the inductive property holds for n + 1, completing the inductive argument.

EXERCISES

11–1. By a *Restricted Conjunctive Sentence,* I mean one which is either an atomic sentence or is a conjunction of an atomic sentence with another restricted conjunctive sentence. Thus the sentences 'A' and '[C&(A&B)]&D' are restricted conjunctive sentences. The sentence 'A &[(C&D)&(H&G)]' is not, because the component, '(C&D)&(H&G)', fails to be a conjunction one of the components of which is an atomic sentence letter.

Here is a rigorous definition of this kind of sentence:

a) Any atomic sentence letter is a restricted conjunctive sentence.
b) Any atomic sentence letter conjoined with another restricted conjunctive sentence is again a restricted conjunctive sentence.
c) Only such sentences are restricted conjunctive sentences.

Such a definition is called an *Inductive Definition.*
Use weak induction to prove that a restricted conjunctive sentence is true iff all the atomic sentence letters appearing in it are true.
11–2. Prove that the formula

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2$$

is correct for all n.

11–3. STRONG INDUCTION

Let's drop the restriction in exercise 11–1 and try to use induction to show that any sentence in which '&' is the only connective is true iff all its atomic sentence letters are true. We restrict attention to any sentence logic sentence, **X**, in which '&' is the only connective, and we do an induction on the number, n, of occurrences of '&'. If n = 0, **X** is atomic, and is true iff all its atomic sentence letters (namely, itself) are true. Next, let's assume, as inductive hypothesis, that any sentence, **X**, in which there are exactly n occurrences of '&' is true iff all its atomic sentence letters are true. You should try to use the inductive hypothesis to prove that the same is true of an arbitrary sentence, **Y**, with n + 1 occurrences of '&'.

If you think you succeeded, you must have made a mistake! There is a problem here. Consider, for example, the sentence '(A&B)&(C&D)'. It has three occurrences of '&'. We would like to prove that it has the inductive property, relying on the inductive hypothesis that all sentences with two

occurrences of '&' have the inductive property. But we can't do that by appealing to the fact that the components, '(A&B)' and '(C&D)', have the inductive property. The inductive hypothesis allows us to appeal only to components which have two occurrences of '&' in them, but the components '(A&B)' and '(C&D)' have only one occurrence of '&' in them.

The problem is frustrating, because in doing an induction, by the time we get to case n, we have proved that the inductive property also holds for **all** previous cases. So we should be able to appeal to the fact that the inductive property holds, not just for n, but for all previous cases as well. In fact, with a little cleverness one can apply weak induction to get around this problem. But, more simply, we can appeal to another formulation of mathematical induction:

> *Weak Induction, Strong Formulation:* Exactly like weak induction, except in the inductive step assume as inductive hypothesis that **P**(i) holds for all i ≤ n, and prove that **P**(n + 1).

EXERCISE

11–3. Using the strong formulation of weak induction, prove that any sentence logic sentence in which '&' is the only connective is true iff all its atomic sentence letters are true.

You could have done the last problem with yet another form of induction:

> *Strong Induction:* Suppose that an inductive property, **P**(n), is defined for n = 1, 2, 3, ... Suppose that for arbitrary n we use, as our inductive hypothesis, that **P**(n) holds for all i < n; and from that hypothesis we prove that **P**(n). Then we may conclude that **P**(n) holds for all n from n = 1 on.

If **P**(n) is defined from n = 0 on, or if we start from some other value of n, the conclusion holds for that value of n onward.

Strong induction looks like the strong formulation of weak induction, except that we do the inductive step for all i < n instead of all i ≤ n. You are probably surprised to see no explicit statement of a basis step in the statement of strong induction. This is because the basis step is actually covered by the inductive step. If we are doing the induction from n = 1 onward, how do we establish **P**(i) for all i < 1? There aren't any cases of i < 1! When n = 1, the inductive hypothesis holds vacuously. In other words, when n = 1, the inductive hypothesis gives us no facts to which to appeal. So the only way in which to establish the inductive step when n = 1 is just to prove that **P**(1). Consequently, the inductive step really covers the case of the basis step. Similar comments apply if we do the induction from n = 0 onward, or if we start from some other integer.

You may be wondering about the connections among the three forms of induction. Weak induction and weak induction in its strong formulation are equivalent. The latter is simply much easier to use in problems such as the last one. Many textbooks use the name 'strong induction' for what I have called 'weak induction, strong formulation'. This is a mistake. Strong induction is the principle I have called by that name. It is truly a stronger principle than weak induction, though we will not use its greater strength in any of our work. As long as we restrict attention to induction on the finite integers, strong and weak induction are equivalent. Strong induction shows its greater strength only in applications to something called "transfinite set theory," which studies the properties of mathematical objects which are (in some sense) greater than all the finite integers.

Since, for our work, all three principles are equivalent, the only difference comes in ease of use. For most applications, the second or third formulation will apply most easily, with no real difference between them. So I will refer to both of them, loosely, as "strong induction." You simply need to specify, when doing the inductive step, whether your inductive hypothesis assumes **P**(i) for all i < n, on the basis of which you prove **P**(n), or whether you assume **P**(i) for all i ≤ n, on the basis of which you prove **P**(n + 1). In either case, you will, in practice, have to give a separate proof for the basis step.

I should mention one more pattern of argument, one that is equivalent to strong induction:

> *Least Number Principle:* To prove that **P**(n), for all integers n, assume that there is some least value of n, say m, for which **P**(m) fails and derive a contradiction.

The least number principle applies the reductio argument strategy. We want to show that, for all n, **P**(n). Suppose that this is not so. Then there is some collection of values of n for which **P**(n) fails. Let m be the least such value. Then we know that for all i < m, **P**(i) holds. We then proceed to use this fact to show that, after all, **P**(m) must hold, providing the contradiction. You can see that this form of argument really does the same work as strong induction: We produce a general argument, which works for any value of m, which shows that if for all i < m **P**(i) holds, then **P**(m) must hold also.

You will notice in exercises 11–7 to 11–9 that you are proving things which in the beginning of Volume I we simply took for granted. Again, this illustrates how some things we take for granted really turn on mathematical induction.

EXERCISES

11–4. Prove that any sentence logic sentence in which 'v' is the only connective is true iff at least one of its atomic sentence letters is true.

11–5. Consider any sentence logic sentence, **X**, in which '&' and 'v' are the only connectives. Prove that for any such sentence, there is an interpretation which makes it true and an interpretation which makes it false. Explain how this shows that '&' and 'v', singly and together, are not expressively complete for truth functions, as this idea is explained in section 3–4, (volume I).

11–6. Consider any sentence logic sentence, **X**, in which '~' does not appear (so that '&', 'v', '⊃', and '≡' are the only connectives). Prove that for any such sentence there is an interpretation which makes **X** true. Explain how this shows that '&', 'v', '⊃', and '≡' are, singly and together, not expressively complete for truth functions.

11–7. Prove for all sentence logic sentences, **X**, and all interpretations, **I**, that either **I** makes **X** true or **I** makes **X** false, but not both.

11–8. Prove for all sentence logic sentences, **X**, that if two truth value assignments, **I** and **I'**, agree on all the atomic sentence letters in **X**, then **I** and **I'** assign **X** the same truth value.

11–9. Prove the law of substitution of logical equivalents for sentence logic.

CHAPTER CONCEPTS

In reviewing this chapter, be sure you understand clearly the following ideas:

a) Weak Induction
b) Inductive Property
c) Basis Step
d) Inductive Hypothesis
e) Inductive Step
f) Induction on the Number of Connectives
g) Strong Formulation of Weak Induction
h) Strong Induction
i) Least Number Principle