Lecture 8

Logical equivalence continued

Special logical equivalences

Introduce logical constants we write as $\underline{t}$ (always true) and $\underline{f}$ (always false). Treat $\underline{t}$ and $\underline{f}$ as new atoms.

$$\boxed{\begin{aligned} p \lor \underline{t} &\equiv \underline{t} \\ p \lor \underline{f} &\equiv p \\ p \land \underline{t} &\equiv p \\ p \land \underline{f} &\equiv \underline{f} \end{aligned}}$$

A wff A is a tautology iff $A \equiv \underline{t}$.

Recall that $A \equiv B$ means that A and B have the same truth tables. The easiest way to show that $A \equiv B$ is to show that $\models A \leftrightarrow B$ using a truth table for $A \leftrightarrow B$.

$\wedge$ and $\vee$ alone

$\wedge$	$\vee$
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
Associativity	associative
$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee r$
Commutativity	Commutative
$p \wedge p \equiv p$	$p \vee p \equiv p$
Idempotence	

$\wedge$ and $\vee$ Combined

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Distributivity | distributive

Negation done

$$\neg\neg p \equiv p$$
double negation

Negation, $\vee$, $\wedge$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
de Morgan's laws

$$p \wedge \neg p \equiv \underline{f}$$
$$p \vee \neg p \equiv \underline{t}$$

Absorption laws

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

<u>All</u> of these special logical equivalences can be checked using truth tables.

<u>Example</u> We prove that $p \vee (p \wedge q) \equiv p$.

We do this by showing $\models p \leftrightarrow (p \vee (p \wedge q))$

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \leftrightarrow (p \vee (p \wedge q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

Thus $\models p \leftrightarrow (p \vee (p \wedge q))$ so

$p \equiv p \vee (p \wedge q)$ as claimed.

We can use known logical equivalences to prove new ones. This can be faster than using truth tables but requires ingenuity.

We use $\boxed{x \to y \equiv \neg x \vee y \quad (*) \text{ repeatedly below}}$

<u>Example 1</u> Prove $p \equiv p \wedge (q \vee \neg q)$ using known logical equivalences

	Reasons
$p \wedge (q \vee \neg q) \equiv p \wedge t$	since $\models q \vee \neg q$
$\equiv p$	since $p \wedge t \equiv p$

"Usually" start with the more complicated wff and simplify it.

$\boxed{\text{Observe that if } A \equiv B \text{ and } B \equiv C \text{ then } A \equiv C}$

Example 2 Prove that $p \to q \equiv \neg q \to \neg p$ using known logical equivalences.

$$
\begin{aligned}
\neg q \to \neg p &\equiv \neg(\neg q) \vee (\neg p) && \text{by } (*) \\
&\equiv q \vee \neg p && \text{double negation} \\
&\equiv \neg p \vee q && \text{Commutativity of } \wedge \\
&\equiv p \to q && \text{by } (*) \text{ again}
\end{aligned}
$$

Example 3 Prove that

$(p \to q) \to q \equiv p \vee q$ using known logical equivalences.

$$
\begin{aligned}
(p \to q) \to q &\equiv \neg(p \to q) \vee q && \text{by } (*) \\
&\equiv \neg(\neg p \vee q) \vee q && \text{by } (*) \\
&\equiv (\neg\neg p \wedge \neg q) \vee q && \text{by De Morgan} \\
&\equiv (p \wedge \neg q) \vee q && \text{by double negation} \\
&\equiv (p \vee q) \wedge (\neg q \vee q) && \text{by distributivity} \\
&\equiv (p \vee q) \wedge T \equiv p \vee q && \models \neg q \vee q,\ x \wedge T \equiv x.
\end{aligned}
$$

<u>Example 4</u> Prove that $p \to (q \to r) \equiv (p \wedge q) \to r$ using known logical equivalences.

$$
\begin{aligned}
p \to (q \to r) &\equiv \neg p \vee (q \to r) && \text{by } (*) \\
&\equiv \neg p \vee (\neg q \vee r) && \text{by } (*) \\
&\equiv (\neg p \vee \neg q) \vee r && \text{associativity} \\
&\equiv \neg (p \wedge q) \vee r && \text{De Morgan} \\
&\equiv (p \wedge q) \to r && \text{by } (*).
\end{aligned}
$$

Example 5 Prove that $p \to (q \to r) \equiv q \to (p \to r)$ using known logical equivalences.

$$
\begin{aligned}
p \to (q \to r) &\equiv \neg p \vee (q \to r) && (*) \\
&\equiv \neg p \vee (\neg q \vee r) && (*) \\
&\equiv \neg q \vee (\neg p \vee r) && \text{Associativity and Commutativity} \\
&\equiv q \to (\neg p \vee r) && (*) \\
&\equiv q \to (p \to r) && (*)
\end{aligned}
$$

Example 6 Prove that $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ using known logical equivalences.

$$
\begin{aligned}
(p \to q) \wedge (p \to r) &\equiv (\neg p \vee q) \wedge (\neg p \vee r) && (*) \\
&\equiv \neg p \vee (q \wedge r) && \text{distributivity} \\
&\equiv p \to (q \wedge r). && (*)
\end{aligned}
$$

<u>Example</u> prove $\models p \to (q \to p)$ using known logical equivalences.

$$p \to (q \to p) \equiv \neg p \vee (\neg q \vee p)$$

$$\equiv \cancel{(\neg p \vee \neg q)} \; (\neg p \vee p) \vee \neg q$$

$$\equiv \underline{t} \vee \neg q$$

$$\equiv \underline{t}$$

$\therefore \; \models p \to (q \to p)$