Lecture 28

We compare the following two sentences in different interpretations.

① $(\exists x)(\forall y)\, F(x,y)$.

② $(\forall y)(\exists x)\, F(x,y)$.

Interpretation #1

D = all people, $F(x,y)$ is interpreted as 'x is the father of y'.

① There is someone who is the father of everyone (F).

② Everyone has a father (T).

Interpretation #2

$D = \mathbb{N} = \{0,1,2,\dots\}$

$F(x,y) = $ '$x \leq y$'

① says, $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})\,(x \leq y)$.

This is T since ~~[illegible]~~ $(\forall y \in \mathbb{N})\,(0 \leq y)$.

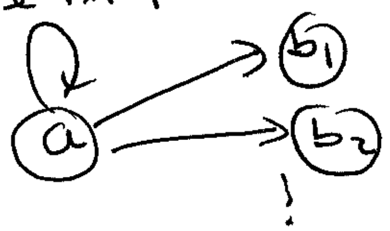
② says $(\forall y \in \mathbb{N})(\exists x \in \mathbb{N})\,(x \leq y)$

~~[illegible]~~ There is an element less than or equal to each element.

This is true since $y \leq y$ is always true.

Can we find an interpretation in which ① is T but ② is F? The answer is <u>no</u>.

S'pose ① is true in some interpretation. Then there is an element $a \in D$ s.t.



But then ② is also true since given any element $b \in D$ there's an element $\bigcirc \to b$ namely a!

It follows that we have proved the following:

$$\boxed{\models (\exists x)(\forall y)F(x,y) \to (\forall y)(\exists x)F(x,y)}$$

[$A \to B$ can only be F when A is T and B is F and we have ruled out that possibility above]

Example We show that the following is a valid argument.

(1). All men are mortal.

(2) Socrates is a man.

(3) ∴ Socrates is mortal.

The form of this argument is as follows:

$$\begin{cases} (1) & (\forall x)\,(H(x) \rightarrow M(x)) \\ (2) & H(a). \\ (3) & \therefore\ M(a). \end{cases}$$

If $(\forall x)(H(x) \rightarrow M(x))$ is true in an interpretation

then $H(a) \rightarrow M(a)$ must be true.

But $H(a)$ is true.

Thus $M(a)$ is true.

We shall extend truth trees to FOL.

- To prove $\models X$ (now, this means X is universally valid) show that the truth tree for $\neg X$ closes.

- To prove $X_1, \ldots X_n \models Y$ show that the truth tree to

$$\begin{array}{c} X_1 \\ \vdots \\ X_n \\ \neg Y \end{array}$$

closes.

- To prove $X \equiv Y$ show that ~~the~~ some truth tree

$$\begin{array}{cc} \multicolumn{2}{c}{\neg(X \leftrightarrow Y)} \\ X & \neg X \\ \neg Y & Y \end{array}$$

closes.

However, our truth trees will not be algorithmic so you will have to use intelligence in applying them rather than just mechanistically applying the rules.

Alan Turing proved that there is no algorithm to decide whether X is universally valid or not thus resolving the Entscheidungsproblem in the negative.

Truth tree rules for FOL

We use the notation $A[x]$ to mean that $A[x]$ is a wff that ~~might~~ contains the variable x. If $(\forall x)A[x]$ is a sentence and we replace all free occurrences of x in $A[x]$ by a parameter a we get $A[a]$ and say that we have *instantiated the universal quantifier at a*. There is a similar procedure for existential quantification.

The leading idea in what follows is this: *convert FOL sentences into PL wff by means of instantiation.*

- All PL truth tree rules are carried forward. OK
- De Morgan's rules for quantifiers:

$$\checkmark\ \neg(\forall x)A \qquad\qquad \checkmark\ \neg(\exists x)A$$
$$| \qquad\qquad\qquad\qquad |$$
$$(\exists x)\neg A \qquad\qquad (\forall x)\neg A$$

- New-name rule.

$$(1)\ (\exists x)A[x]\checkmark$$
$$|$$
$$A[a]$$

where we add $A[a]$ at the bottom of all branches containing (1) and where a is a ~~parameter~~ that does not already appear in the branch containing (1). Constant

- Never-ending rule.

$$(2)\ (\forall x)A[x]\ *$$
$$|$$
$$A[a]$$

where we add $A[a]$ at the bottom of a branch containing (2) and a is any ~~parameter~~ appearing in the branch containing (2) or a is a new ~~parameter~~ if no ~~parameters~~ have yet been introduced. [The rationale for the latter is that all domains are non-empty]. We have used the $*$ to mean that the wff is never used up. Constant constants

<u>Example</u> We prove using truth trees

that

$$\models \quad (\exists x)(\forall y)F(x,y) \rightarrow (\forall y)(\exists x)F(x,y)$$

$$\neg\,[\;\underline{(\exists x)(\forall y)\,F(x,y)} \rightarrow \underline{(\forall y)(\exists x)F(x,y)}\;] \checkmark$$

$$|$$

$$(\exists x)(\forall y)\,F(x,y)$$

$$\neg\,(\forall y)(\exists x)\,F(x,y) \checkmark$$

Tree closes so wff is universally valid

new name rule

$$|$$

$$(\exists y)(\forall x)\,\neg F(x,y)$$

$$|$$

$$(\forall y)\,F(a,y)\ *$$

new name rule

$$|$$

$$(\forall x)\,\neg F(x,b)\ *$$

$$|$$

$$F(a,b)$$

$$\neg F(a,b)\ \times$$