Lecture 27

The syntax of FOL

This consists of a choice of predicate symbols. In addition,

PL: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \oplus$

Variables: $x_1, x_2, x_3, \ldots$

Constants: $a_1, a_2, a_3, \ldots$

Quantifiers: $(\forall x_1), (\forall x_2), \ldots$
$(\exists x_1), (\exists x_2), \ldots$

Atomic formulae: Predicate letters with variables and constants in all available slots.

Formulae/wff

(F1) All atomic formulae are formulae.

(F2) If A and B are formulae so too are

$(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $(A \oplus B)$,

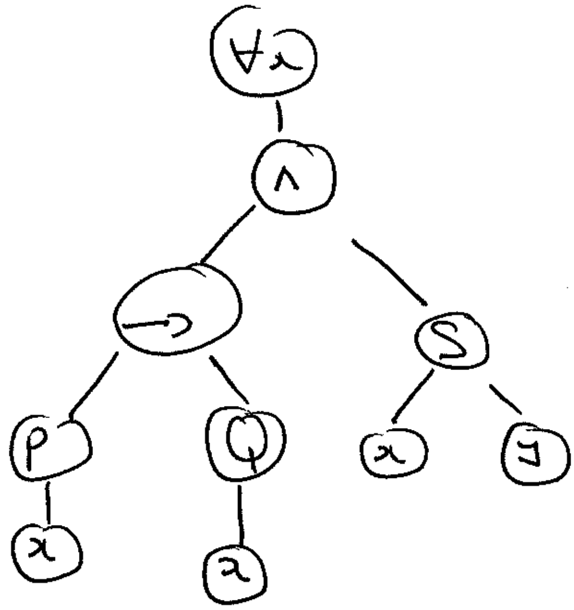
$(\forall x) A$, $(\exists x) B$

any variable

(F3) All formulae are obtained by a finite number of applications of (F1) and (F2).

<u>Example</u> We draw the parse tree of the formula

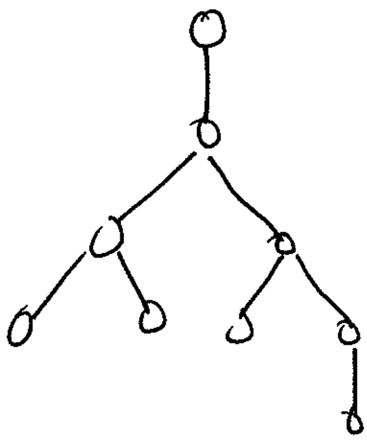
$$(\forall x)\left(\left(P(x) \to Q(x)\right) \wedge S(x,y)\right)$$



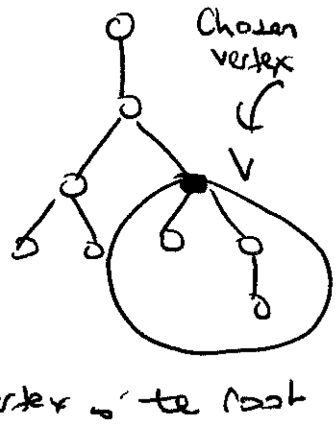
Unlike PL, we shall not study all formulae. We shall focus on those we call <u>sentences</u>. This is a technical term which I shall now explain.

<u>subtrees</u>

tree →



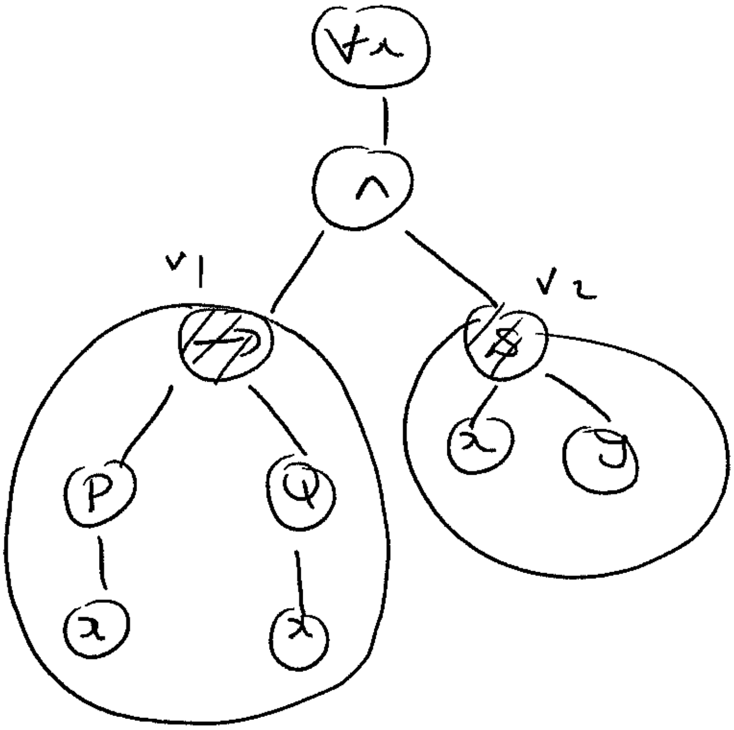
choose any vertex which is not a leaf



The chosen vertex is the root of a tree called <u>the subtree determined by v</u>

The idea of a subtree leads to the definition of a <u>subformula</u>.

Example



subformula det'd by v_1 is $P(x) \to Q(x)$

subformula det'd by v_2 is $S(x, y)$

Key definition Let A be a formula. Let $(\forall x)$ (or $(\exists x)$) be an occurrence of $(\forall x)$ (or $(\exists x)$) in A. This determines a subformula of A, $(\forall x)B$ (or $(\exists x)B$). An occurrence of the variable x in $(\forall x)B$ (or $(\exists x)B$) is said to be bound.

A sentence is a formula in which every variable is bound.

You should regard constants as being automatically bound.

Examples

(1) $(\forall x)\big((P(x) \to Q(x)) \wedge S(x, y)\big)$

is not a sentence since y is not bound (= free).

(2) $(\forall x)(\forall y)\big(P(x) \to Q(y)\big)$

is a sentence because every occurrence of every variable is bound.

(3) $(\exists x)\big(F(x) \wedge G(x)\big) \to \big((\exists x)F(x) \wedge (\exists x)G(x)\big)$

This is a sentence because every occurrence of a variable is bound.

Semantics of FOL

Let L be a first order language. For example, suppose it consists of P 1-place predicate symbol
Q 2-place predicate symbol.

An interpretation of L is any structure (D, A, ρ) where $A \subseteq D$ and $\rho \subseteq D^2$ [D^2 = set of all ordered pairs from D] a binary relation. We interpret P as A and Q as ρ. Thus, for example, the formula $(\exists x)(P(x) \wedge Q(x,x))$

is interpreted as

$$(\exists x)\big((x \in A) \wedge (x,x) \in \rho\big)$$

Under an interpretation every sentence S in L makes an assertion about the structure.
If S is true in this interpretation, we say it is a model of S.
If S is true in all interpretations, we write

$\models S$ and say S is universally valid.

[FOL studies universally valid formulae]

Why sentences?

Consider the 2-place predicate

$$F(x,y) = \text{'}x \text{ is the father of } y\text{'}$$

is neither true or false

$$(\forall y)\, F(x,y)$$

'for each person, x is their father' is neither true or false

~~fine~~: $$(\exists x)(\forall y)\, F(x,y)$$

'There is someone who is the father of everyone' (false).

Now compare this with the formula

$$(\forall y)(\exists x)\, F(x,y)$$

'Everybody has a father' (true)

(So, the order of quantifiers will be important).

> Sentences when interpreted in a structure become either true or false