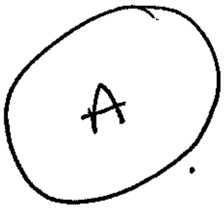
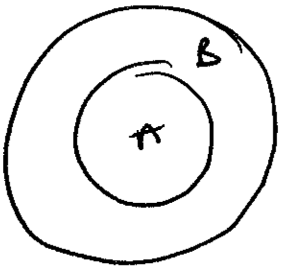
Lecture 22

Venn diagrams

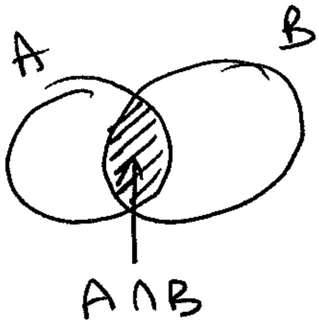
We picture a set A as a region in the plane





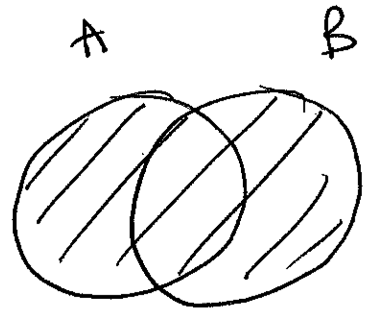
says that $A \subseteq B$.

Boolean operations $\cap, \cup, \setminus$



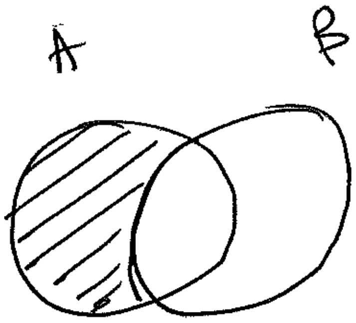
$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$

Called the <u>intersection</u> of A and B



$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

the union of A and B



$$A \setminus B = \{x : (x \in A) \wedge \neg (x \in B)\}$$

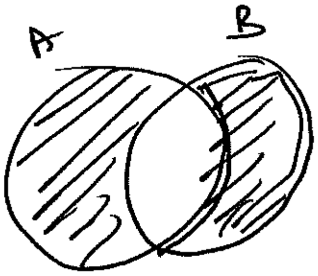
~~difference~~

Set difference.

Example

Draw the set

$$\{x : (x \in A) \oplus (x \in B)\}$$



Example $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$A \cap B = \{3, 4\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

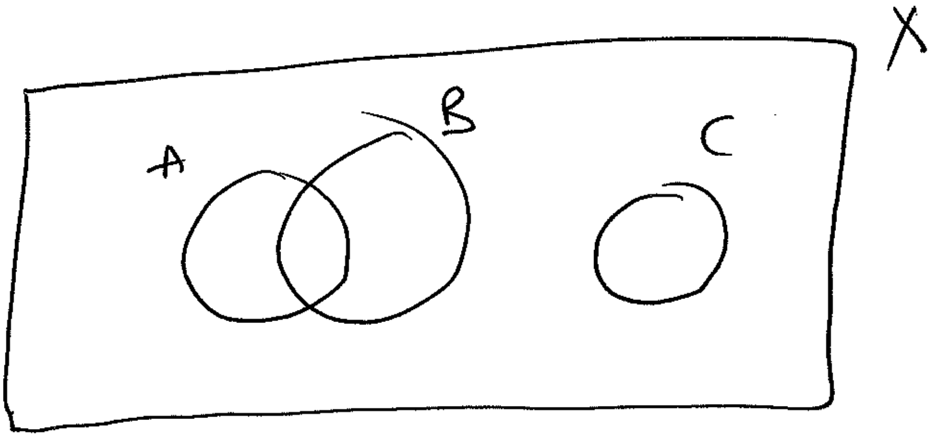
$A \setminus B = \{1, 2\}$

$B \setminus A = \{5, 6\}$

Important construction

Let X be a fixed set (called the *frame*)

Let $A \subseteq X$. Define $\overline{A} = X \setminus A$.



We now describe the properties of the operations $\cap, \cup, \overline{}$ on the power set $P(X)$.

F

Union

(1) $(A \cup B) \cup C = A \cup (B \cup C)$.

(2) $A \cup B = B \cup A$.

(3) $A \cup \emptyset = A$.

Intersection

(4) $(A \cap B) \cap C = A \cap (B \cap C)$.

(5) $A \cap B = B \cap A$.

(6) $A \cap X = A$.

Both

(7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Complementation

(9) $A \cup \overline{A} = X$.

(10) $A \cap \overline{A} = \emptyset$.

$\boxed{\text{We can prove all three equations by using logical equivalences.}}$

Example $(p \vee q) \vee r \equiv p \vee (q \vee r)$

implies $(A \cup B) \cup C = A \cup (B \cup C)$.

Put

$p = {}'x \in A'$, $q = {}'x \in B'$, $r = {}'x \in C'$.

Then $x \in (A \cup B) \cup C$ iff

$$((x \in A) \vee (x \in B)) \vee (x \in C)$$

$$\equiv (x \in A) \vee ((x \in B) \vee (x \in C))$$

iff $x \in A \cup (B \cup C)$.

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observe $'x \in \emptyset'$ is always F.

$'x \in X'$ is always T.

Table of Correspondences

PL	Sets (subsets of X)
$\equiv$	$=$
$\vee$	$\cup$
$\wedge$	$\cap$
$\neg$	$-$
$\underline{f}$	$\emptyset$
$\underline{t}$	X

PL logical equivalences

(1) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

(2) $p \vee q \equiv q \vee p$.

(3) $p \vee \underline{f} \equiv p$.

(4) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

(5) $p \wedge q \equiv q \wedge p$.

(6) $p \wedge \underline{t} \equiv p$.

(7) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

(8) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(9) $p \vee \neg p \equiv \underline{t}$.

(10) $p \wedge \neg p \equiv \underline{f}$.

Boolean algebra

$$(B,\ \underbrace{+,\ \cdot}_{\text{binary}},\ \underbrace{\bar{\ }}_{\text{unary}},\ \underbrace{0,\ 1}_{\text{constants.}})$$

(B1) $(x+y)+z = x+(y+z)$.

(B2) $x+y = y+z$.

(B3) $x+0 = x$.

$\boxed{x \cdot y = xy \text{ usual notation.}}$

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$

(B6) $x \cdot 1 = x$.

(B7) $x \cdot (y+z) = x \cdot y = x \cdot z$.

(B8) $x + (y \cdot z) = (x+y) \cdot (x+z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

Examples

(1) $\left(P(X), \cap, \cup, \bar{\ }, \emptyset, X\right)$ is a BA.

$$\boxed{\begin{array}{l}(2)\ \left(\mathbb{B}=\{0,1\}, +, \cdot, \bar{\ }, 0, 1\right) \\ \text{2-element BA} = P(\{1\}). \\ \text{used in circuit design.}\end{array}}$$

We look at example (2) the 2-element Boolean algebra.

$$P(\{1\}) = \{\emptyset, \{1\}\}$$

$$\overline{\emptyset} = \{1\} \quad \wedge \quad \overline{\{1\}} = \emptyset$$

$\cap$	$\emptyset$	$\{1\}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{1\}$

$\cup$	$\emptyset$	$\{1\}$
$\emptyset$	$\emptyset$	$\{1\}$
$\{1\}$	$\{1\}$	$\{1\}$

Write $\emptyset \to 0$ and $\{1\} \to 1$

Then we get the following Boolean operations on the set $\mathbb{B} = \{0, 1\}$

a	$\overline{a}$
0	1
1	0

$a \cdot b$ $\parallel$

a	b	$a \cap b$
1	1	1
1	0	0
0	1	0
0	0	0

$a+b$ $\parallel$ ← Boolean addition **NOT** the usual addition.

a	b	$a \cup b$
1	1	1
1	0	1
0	1	1
0	0	0

The 2-element Boolean algebra $\mathbb{B} = \{0,1\}$ with operations as defined above is the basis of all digital circuit design.