Lecture 19

Truth trees Continued

Review of truth tree rules.

The goal of a truth tree is to show that the wff at the root of the tree is satisfiable.

If the truth tree closes — that is, every branch is closed — then the wff is a contradiction.

We say that a truth tree is finished if it is closed or no further rules can be applied.

Example

$$\begin{array}{c} p \wedge \neg p \\ | \\ p \\ \neg p \\ \times \end{array}$$

$\therefore$ $p \wedge \neg p$ is a contradiction.

To show that $\models X$ prove that

$\neg X \models$

<u>Example</u> $\models p \vee \neg p$.

$$\neg(p \vee \neg p) \checkmark$$
$$|$$
$$\neg p$$
$$\neg\neg p \checkmark$$
$$|$$
$$p$$
$$\times$$

Truth table closes. ∴ $\neg(p \vee \neg p)$ is a contradiction and so $p \vee \neg p$ is a tautology.

<u>Example</u> Prove that

$$\models \overbrace{((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))}^{X}$$

$$\neg X =$$

$$\neg\left[\, ((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))\,\right] \checkmark$$

$$(p \to q) \wedge (p \to r) \checkmark$$
$$\neg(p \to (q \wedge r)) \checkmark$$

$$p \to q \checkmark$$
$$p \to r$$

$$p$$
$$\neg(q \wedge r) \checkmark$$

$\neg p$ × | q

$\neg q$ × | $\neg r$

$\neg p$ × | r ×

truth tree closes so X is a tautology.

Example Prove that

$$\models (p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

$$\neg[(p \to (q \to r)) \to ((p \to q) \to (p \to r))] \checkmark$$

$$p \to (q \to r)$$

$$\neg[(p \to q) \to (p \to r)] \checkmark$$

$$p \to q \checkmark$$

$$\neg(p \to r) \checkmark$$

Truth table closes

$$p$$

$$\neg r$$

$\neg p$ $\times$ | q

$\neg p$ $\times$ | $q \to r$

$\neg q$ $\times$ | r $\times$

We now ~~show~~ explain how to use truth trees to show that an argument is valid.

$X_1, \ldots X_n \models Y$ precisely when

$$\models (X_1 \wedge \ldots \wedge X_n) \to Y$$

So, we put $\neg[(X_1 \wedge \ldots \wedge X_n) \to Y]$ at the root of the tree and show the tree closes.

$$\neg[(X_1 \wedge \ldots \wedge X_n) \to Y] \equiv \neg[\neg(X_1 \wedge \ldots \wedge X_n) \vee Y]$$

$$\equiv \neg\neg(X_1 \wedge \ldots \wedge X_n) \wedge \neg Y$$

$$\equiv X_1 \wedge \ldots \wedge X_n \wedge \neg Y$$

Put

$$\boxed{\begin{array}{c} X_1 \\ \vdots \\ X_n \\ \neg Y \end{array}}$$

at the root of the tree.

Example Show that the argument $p, p \to q \models q$ is valid

p

$p \to q$ ✓

$\boxed{\neg q}$

$\neg p$	q
×	×

Truth tree closes so argument is valid.

Example $p \to q, \neg q \models \neg p$

$p \to q$

$\neg q$

$\boxed{\neg\neg p}$ ✓

p

$\neg p$	q
×	×

3.

Example $\quad p \vee q, \neg p \models q$

$p \vee q$ ✓

$\neg p$

$\boxed{\neg q}$

$p \quad\quad q$

× $\quad\quad$ ×

Example $\quad p \to q, \; q \to r \models p \to r$

$p \to q$ ✓

$q \to r$

$\boxed{\neg(p \to r)}$ ✓

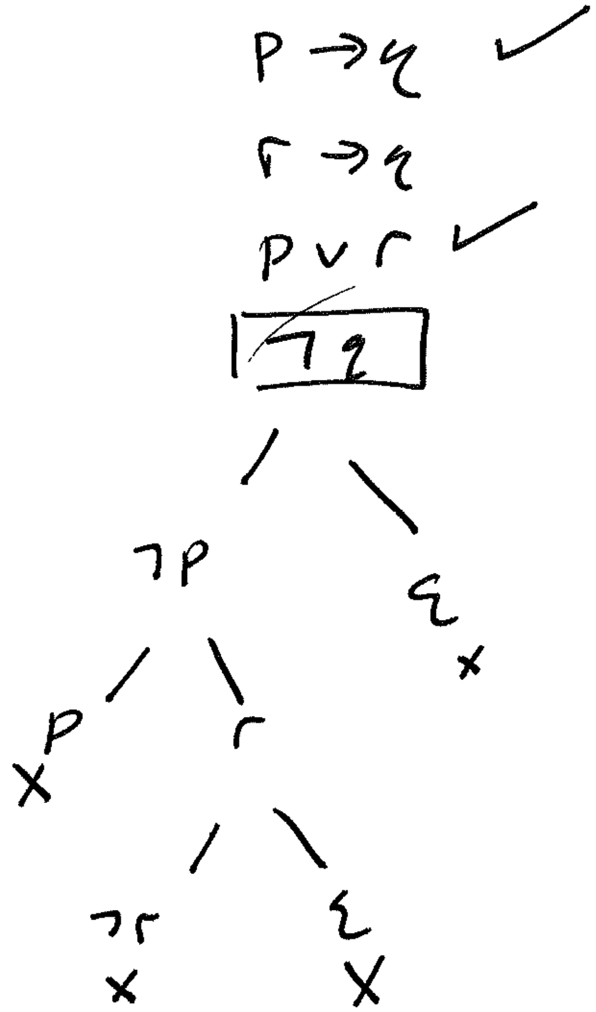
p

$\neg r$

$\neg p$ × $\quad\quad q$

× $\neg q \quad\quad r$ ×

<u>Example</u> $\quad p \to q,\ \ r \to q \quad p \vee r \models q$



The theory of truth trees

This is discussed in more detail in section 1.11.2 of the book. I shall just highlight the important ideas here.

Result 1 If a finished truth tree for X has an open branch then X is satisfiable.

The set of wff on an open branch of a finished truth tree is an example of what is called a **Hintikka set**. Assigning truth values to the literals occurring in a Hintikka set (T if literal is +ve; F if literal is −ve) leads to a truth assignment making all wff in the set true.

<u>Result 2</u> If X is a wff and τ is any truth assignment satisfying X then any finished truth tree for X has an open branch s.t. all wff in that branch are true under τ.

<u>Theorem</u> (Soundness and Completeness)

<u>(1) Soundness.</u> If X is a tautology then every truth tree with root $\neg X$ is closed.

<u>(2) Completeness.</u> If some finished truth tree for $\neg X$ is closed then X is a tautology.

<u>Proof</u> (1). S'pose there is a finished truth tree for $\neg X$ which is not closed. Then it has an open branch. By Result 1, $\neg X$ is satisfiable and so X is not a tautology. Contradiction ∎

(2) Let T be a finished truth tree for $\neg X$ which is closed. If X were not a tautology then $\neg X$ would be satisfiable. Let τ be a truth assignment so that $\tau(\neg X) = T$. By Result 2, T would have an open branch along which τ is true. Contradiction. Thus X is a tautology $\blacksquare$