Lecture 16

1.10 Valid arguments

We now come to the raison d'être of PL (and logic in general): the formalization of arguments.

Examples

(1) I will either drink cider or apple juice.
I will not drink cider.
$\therefore$ I will drink apple juice.

The argument has the following form (the argument is valid not because of its particulars but because of its form)

$$\boxed{p \vee q,\ \neg p\ \therefore\ q}$$

Let's see why this argument is valid.
Suppose that $p \vee q$ and $\neg p$ are true.
We shall prove that q must be true.

$p \vee q$ true means either p is true or q is true.

$\neg p$ true means that p is false.

$\therefore$ q must be true.

We can also check the validity of this argument using truth tables.

p	q	$p \vee q$	$\neg p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

Now erase all rows except where $p \vee q$ and $\neg p$ are true.

p	q	$p \vee q$	$\neg p$
~~T~~	~~T~~	~~T~~	~~F~~
~~T~~	~~F~~	~~T~~	~~F~~
F	T	T	T
~~F~~	~~F~~	~~F~~	~~T~~

We see that q is true.

(2) Either Smith or Jones will win the election.

If Smith wins we are doomed.

If Jones wins we are doomed.

$\therefore$ We are doomed.

The form of this argument is as follows:

$$\boxed{p \vee q,\ p \rightarrow r,\ q \rightarrow r \ \therefore\ r}$$

Assume $p \vee q$, $p \rightarrow r$, $q \rightarrow r$ are all true.

$p \vee q$ true means either p is true or q is true.

S'pose that p is true.

Then r must be true.

S'pose that q is true.

Then r must be true.

It follows that in both cases r is true.

We can also use truth tables.

p	q	r	$p \vee q$	$p \to r$	$q \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	T
F	F	F	F	T	T

Now erase all rows where $p \vee q$, $p \to r$, $q \to r$ not true.

p	q	r	$p \vee q$	$p \to r$	$q \to r$
T	T	T	T	T	T
~~T~~	~~T~~	~~F~~	~~T~~	~~F~~	~~F~~
T	F	T	T	T	T
~~T~~	~~F~~	~~F~~	~~T~~	~~F~~	~~T~~
F	T	T	T	T	T
~~F~~	~~T~~	~~F~~	~~T~~	~~T~~	~~F~~
~~F~~	~~F~~	~~T~~	~~F~~	~~T~~	~~T~~
~~F~~	~~F~~	~~F~~	~~F~~	~~T~~	~~T~~

We see that r takes the value true in all these cases.

(3) If it is raining then I will get wet.

I am not wet.

∴ It is not raining.

$$\boxed{p \to q,\ \neg q \ \therefore\ \neg p}$$

Suppose $p \to q$ and $\neg q$ are true.

Then q is false.

Thus p is false.

∴ $\neg p$ is true.

p	q	$p \to q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Now erase all rows where $p \to q$, $\neg q$ are not true.

p	q	$p \to q$	$\neg p$	$\neg q$
~~T~~	~~T~~	~~T~~	~~F~~	~~F~~
~~T~~	~~F~~	~~F~~	~~F~~	~~T~~
~~F~~	~~T~~	~~T~~	~~T~~	~~F~~
F	F	T	T	T

∴ $\neg p$ is true.

We want to formulate within PL how the word 'therefore' is being used in each of these arguments.

<u>Definition</u> Let $A_1, \dots A_n, B$ be statements. We say that B <u>follows from</u> $A_1 \dots A_n$, written $A_1, \dots, A_n \models B$ if whenever $A_1 \dots A_n$ are all true then B is also true.

We say that $A_1 \dots A_n \therefore B$ is a <u>valid argument</u>.

<u>Notation</u> I usually read $\models$ as 'therefore'

The following shows the close connection between valid arguments and tautologies.

<u>Theorem</u> $A_1, \ldots, A_n \models B$ is a valid argument precisely when $\models (A_1 \wedge \ldots \wedge A_n) \to B$.

<u>Proof</u> suppose that $A_1 \ldots A_n \models B$ is a valid argument. If $(A_1 \wedge \ldots \wedge A_n) \to B$ is not a tautology then there is some assignment of truth values to the atoms such that $A_1 \wedge \ldots \wedge A_n$ is true and B is false. But this contradicts the assumption that $A_1, \ldots A_n \models B$. ■

suppose that $\models (A_1 \wedge \ldots \wedge A_n) \to B$.

Suppose all $A_1 \ldots A_n$ are true. Then $A_1 \wedge \ldots \wedge A_n$ is true thus B is true. ■

Examples Prove the following

(1) $\models ((p \vee q) \wedge \neg p) \rightarrow q$.

(2) $\models ((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$.

(3) $\models ((p \rightarrow q) \wedge (\neg q)) \rightarrow \neg p$.

Not all valid arguments can be formalized within PL.

Example All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

This is a valid argument but cannot be formalized within PL. To do this, we need first-order logic (FOL).

p	q	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	T
T	F	T
F	T	T
F	F	T

expression is a **tautology**

p	q	r	$((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

expression is a **tautology**

p	q	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
T	T	T
T	F	T
F	T	T
F	F	T

expression is a **tautology**