Lecture 13

Section 1.8 Continued...

Background to PROLOG

Determining whether a wff is satisfiable or not usually involves a lot of work (using truth tables, for example). However, some wff have a shape that means determining whether they are satisfiable or not is very simple (and does not require their truth table to be constructed). Such wff might be effectively useless (so, who cares if they are satisfiable or not), but it turns out that there is a class of wff whose satisfiability is easy to determine and which are useful. As it happens, these wff form the basis of PROLOG (at least, when generalized to FOL)

Recall that we defined the logical constants $\underline{t}$ and $\underline{f}$ where $\underline{t}$ is always true and $\underline{f}$ is always false.

- Observe that $\underline{f} \vee p \equiv p$.

But $\underline{f} \equiv \neg \underline{t}$. Thus

$$\neg \underline{t} \vee p \equiv p.$$

We can write this as

$$\boxed{\underline{t} \rightarrow p \equiv p}$$

- Clearly, $\underline{f} \vee \neg p \equiv \neg p$.

But $\underline{f} \vee \neg p \equiv \neg p \vee \underline{f} \equiv p \rightarrow \underline{f}$.

$$\boxed{p \rightarrow \underline{f} \equiv \neg p}$$

Some terminology

literal = p or $\neg p$

positive literal = atom = p

negative literal = $\neg p$

Recall that a wff is in CNF if it is a conjunction of one or more blocks where each block is a disjunction of one or more literals.

$(\vee \text{ literals}) \wedge (\vee \text{ literals}) \wedge \ldots$

Definition A wff in CNF is called a Horn* formula if each block contains at most one positive literal.

* Named after the Mathematician Alfred Horn.

Example I claim that H below is a Horn formula

$$H = (\neg r \vee \neg s \vee u) \wedge (\neg p \vee q) \wedge (\neg q \vee r) \wedge (\neg r \vee s) \wedge (r) \wedge (\neg p \vee \neg q)$$

Each block contains at most one true literal.

This defn looks completely arbitrary, but it looks better when transformed to <u>implicational form</u>

<u>Example</u>

(1) $\neg r \vee \neg s \vee u \equiv \neg(r \wedge s) \vee u$

$\equiv (r \wedge s) \to u$

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(2) $\neg p \vee q \equiv p \to q.$

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(3) $\neg q \vee r \equiv q \to r.$

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(4) $\neg r \vee s \equiv r \to s$

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(5) $r \equiv \underline{t} \to r$

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(6) $\neg p \vee \neg q \equiv \neg(p \wedge q) \equiv (p \wedge q) \to \underline{f}$

We therefore have that

$$H \equiv [(r \wedge s) \to u] \wedge [p \to q] \wedge [q \to r]$$

$$\wedge [r \to s] \wedge [\underline{t} \to r] \wedge [(p \wedge q) \to \underline{f}]$$

Implicational form of the Horn formula H.

More generally

- $\neg p_1 \vee \dots \vee \neg p_{n-1} \vee p_n$ one +ve literal and at least one -ve literal

$$\equiv \neg(p_1 \wedge \dots \wedge p_{n-1}) \vee p_n$$

$$\equiv (p_1 \wedge \dots \wedge p_{n-1}) \to p_n$$

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one +ve literal and no -ve literal

- $p \equiv \underline{f} \vee p \equiv \neg \underline{t} \vee p \equiv \underline{t} \to p$

- $\neg p_1 \vee \ldots \vee \neg p_n \equiv \neg(p_1 \wedge \ldots \wedge p_n)$ no true literals

$\equiv \neg(p_1 \wedge \ldots \wedge p_n) \vee \underline{f}$

$\equiv (p_1 \wedge \ldots \wedge p_n) \rightarrow \underline{f}$

Proposition (Implicational form)

Every Horn formula is logically equivalent to one which is a conjunction of wff of the following forms:

(1) $\top \to p$.

(2) $(p_1 \wedge \ldots \wedge p_n) \to q$.

(3) $(p_1 \wedge \ldots \wedge p_n) \to \underline{f}$

It turns out that Horn formulae are sufficiently expressive to be useful but, at the same time, there is a very fast algorithm to determine whether they are satisfiable or not.
