Lecture #10

1.6. Adequate sets of connectives

In this section, we shall show that our choice of connectives was (within reason) arbitrary. We begin with some observations.

(1) $p \oplus q \equiv \neg(p \leftrightarrow q)$.

So, we could do without $\oplus$ and use only $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

(2) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

So, we could do without $\leftrightarrow$ as well and use only $\neg, \wedge, \vee, \rightarrow$.

(3) $p \rightarrow q \equiv \neg p \vee q$.

So, we could do without $\rightarrow$ as well and use only $\neg, \wedge, \vee$.

We have therefore proved the following

<u>Proposition</u> Every wff is logically equivalent to one in which the only connectives that appear are $\neg, \vee, \wedge$.

<u>Definition</u> A set S of connectives is said to be <u>adequate</u> if every wff is logically equivalent to a wff constructed using only connectives from S.

Thus: $\boxed{\neg, \wedge, \vee \text{ is an adequate set of connectives.}}$

Can we do better?

By De Morgan's laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\therefore \quad p \wedge q \equiv \neg(\neg p \vee \neg q)$$

<u>Proposition</u> $\neg, \vee$ is an adequate set of connectives.

Also, $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$$\therefore \quad p \vee q \equiv \neg(\neg p \wedge \neg q)$$

<u>Proposition</u> $\neg, \wedge$ is an adequate set of connectives.

Can we get away with only one connective?

Yes, but it will be a new connective — in fact, there are two of them.

<u>Definition</u> $p \ \underline{\text{nand}} \ q \overset{\text{def}}{=} \neg(p \wedge q)$

$p \ \underline{\text{nor}} \ q \overset{\text{def}}{=} \neg(p \vee q).$

<u>Proposition</u>

(1) <u>nand</u> is an adequate set of connectives on its own.

(2) <u>nor</u> is an adequate set of connectives on its own.

Proof

(1) $$p \underline{\text{nand}}\, p \equiv \neg(p \wedge p) \equiv \neg p$$

$$p \underline{\text{nand}}\, q \overset{\text{df}}{=} \neg(p \wedge q)$$

$$\therefore\ p \wedge q \equiv \neg(p \underline{\text{nand}}\, q)$$

$$\equiv (p \underline{\text{nand}}\, q)\, \underline{\text{nand}}\, (p \underline{\text{nand}}\, q)$$

But $\neg, \wedge$ is an adequate set of connectives so nand, on its own is an adequate set of connectives.

(2) The proof for nor is similar and left as an exercise.

The above result has wider significance (& since after we have studied Section 1.7 (and where we are in Chapter 2), it will imply that every circuit can be built from nor-gates (or nand-gates)

Although we could develop PL with a smaller set of connectives than the one we chose (including only one), it would not be very user-friendly to do so.
A balance has to be struck between too many (hard to remember) and too few (not user friendly).

1.7 Truth functions

In this section, we shall show how to go from a truth table to a wff which has this truth table. This result is important in circuit design.

Let $p_1, \ldots, p_n$ be atoms.
Let $q_1, \ldots, q_n$ be literals – so $q_i = p_i$ or $\neg p_i$.

We call $q_1 \wedge \ldots \wedge q_n$ a

<u>Conjunctive clause</u>.

<u>Example</u> $p \wedge q \wedge r$, $\neg p \wedge q \wedge r$, $p \wedge \neg q \wedge \neg r$ etc are all conjunctive clauses.

Conjunctive clauses are very special kinds of wff. Their truth tables are T exactly once.

Examples

(1) $p \wedge q \wedge r$ is True

p	q	r
T	T	T

and F in all other cases.

(2) $\neg p \wedge q \wedge r$ is true

p	q	r
F	T	T

and F in all other cases.

(3) $p \wedge \neg q \wedge \neg r$ is true

p	q	r
T	F	F

and F in all other cases.

We now show how every truth table (or truth function) is the truth table of some wff.

Example

p	q	r	A?
T	T	T	F
T	T	F	F
T	F	T	T ①
T	F	F	T ②
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T ③

Look at the rows where the outputs are T (①, ②, ③)

For each such row, construct a conjunctive clause

where you see F negate the corresponding variable

① $p \wedge \neg q \wedge r$

② $p \wedge \neg q \wedge \neg r$

③ $\neg p \wedge \neg q \wedge \neg r$

Now put

$$A = (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$$
$$\vee\ (\neg p \wedge \neg q \wedge \neg r)$$

<u>Claim</u> The truth table of A is the truth table I started with.

This construction generalizes.

[If the output is always F, then use $(p \wedge \neg p) \wedge q \wedge r$, for example]