

Boolean inverse Semigroups

by

Mark V Lawson

Heriot - Watt University

Edinburgh

February 2020

With the collaboration of :

Johannes Kellendonk, Daniel Lenz,
Ganna Kudryavtseva, Pedro Resende,
Phil Scott, Alina Vdovina, Aidan Sims.

" "

Philosophy

- Need a class of semigroups that is as wide as possible, but not so wide that no interesting theorems can be proved.
- Need good examples.
- Would like natural connections with other parts of mathematics.

Approach

Generalize the commutative structures of lattice theory

frames, distributive lattices, Boolean algebras

to a non-commutative setting

pseudogroups, distributive inverse semigroups

Boolean inverse semigroups.

In this talk, I will only deal with

Boolean inverse semigroups.

These are non-commutative generalizations
of generalized Boolean algebras.

Boolean inverse monoids
generalize Boolean algebras.

A generalized Boolean algebra is a distributive

lattice with zero equipped with a binary operation

✓ satisfying

$$0 = y \wedge (x \vee y) \text{ and } x = (x \vee y) \vee (x \wedge y).$$

A Boolean algebra is a unital generalized Boolean

algebra. This is equipped with a unary operation,

the complement, given by $\bar{e} = 1 \setminus e$.

Ideas (notation from inverse semigroups)

\leq the natural partial order

\sim the compatibility relation

$E(S)$ Semilattice of idempotents

$U(S)$ group of units (when S is a monoid)

$$\underline{d}(a) = \bar{a}'a, \quad \underline{I}(a) = a\bar{a}'$$

Reduced product $a \cdot b (= ab)$ defined when $\underline{d}(a) = \underline{I}(b)$.

Boolean inverse semigroup

- Inverse semigroup S .
- $x \sim y \Leftrightarrow \exists uv \text{ in } S$.
- $\forall \exists xy$ then for any $z \in S$,
 $\exists zxvzy$ and $\exists xzvyz$ and
 $z(xy)z = zxvzy$ and $(xv)z = xzvyz$.
- $\underline{E}(S)$ is a generalized Boolean algebra
 w.r.t. \leq as its order.

Notation in Boolean inverse semigroups

- $\underline{e}(x) = \underline{d}(x) \vee \underline{f}(y)$, the extent of x .
 - $x \perp y$, x is orthogonal to y , if $\underline{d}(x) \underline{d}(y) = 0$
and $\underline{f}(x) \underline{f}(y) = 0$. Their join is denoted by $x \oplus y$.
 - If $y \leq x$ define $x \setminus y = x(\underline{d}(x) \setminus \underline{d}(y))$.
- Observe that $x = y \oplus (x \setminus y)$.

First examples

1. $I(X)$ symmetric inverse monoids
are Boolean inverse monoids. Denote by I_n
the symmetric inverse monoid on n letters.

2. $I^{fin}(X)$ partial bijections with a finite domain.
These are Boolean inverse semigroups.

Important definitions

An inverse semigroup is fundamental if the only elements commuting with all idempotents are idempotents.

An additive ideal in a Boolean inverse semigroup is a semigroup ideal closed under binary joins.

A Boolean inverse semigroup is 0-simplifying if it has only trivial additive ideals.

A Boolean inverse semigroup is simple if it is 0-simplifying and fundamental.

In this talk I shall
focus on Boolean inverse monoids
to simplify the presentation.

Finite Boolean inverse monoids

Remarkably, these can be described explicitly.

Rook matrices

Let S be a Boolean inverse monoid.

An $n \times n$ rook matrix over S is such that elements of S in the same row have orthogonal ranges, and elements of S in the same column have orthogonal domains.

Denote by $M_n(S)$ the set of all $n \times n$ rook matrices over S .

[Goes back to Peter Hines' thesis]

Proposition $M_n(S)$ is a Boolean inverse monoid. Multiplication is like matrix multiplication except addition is replaced by join.

If $A \in M_n(S)$ then its inverse A^* is defined by $A^* = (a_{ji}^{-1})$.

The idempotents of $M_n(S)$ are the diagonal matrices whose diagonal entries are idempotents.

Theorem.

Let S be a finite Boolean inverse monoid.


Then there are finite groups G_1, \dots, G_r

and natural numbers n_1, \dots, n_r s.t.

$$S \cong M_{n_1}(G_1^0) \times \dots \times M_{n_r}(G_r^0).$$

S is fundamental $\Leftrightarrow S \cong I_{n_1} \times \dots \times I_{n_r}$.

S is simple $\Leftrightarrow S \cong I_n$ for some finite n .

Proof Let S be a finite Boolean inverse monoid. Let G be the set of atoms of S . W.r.t. the reduced product G is a groupoid. Now use standard groupoid theorem. The proof generalizes the classical proof giving the structure of finite Boolean algebras. 

The Tarski algebra is the unique countable atomless Boolean algebra

A Boolean inverse monoid will be called a Tarski monoid if it is countably infinite and its semilattice of idempotents is a Tarski algebra.

The Dichotomy Theorem

Let S be a simple, countable Boolean inverse monoid. Then

(i) $S \cong I_n$ for some finite n .

OR

(ii) S is a Tarski monoid.

Under non-Commutative Stone duality,

Tarski monoids correspond to

second-countable étale topological groupoids with a Cantor space of identities; these groupoids are also minimal and effective. They are Hausdorff precisely when the Tarski monoid has all binary meets.

Such groupoids are studied by Hiroki Matui and are of interest to C^* -algebra theorists.

The groups of units of Tarski monoids are countably infinite analogues of finite symmetric groups.

Example We show that the classical Thompson groups arise in this way.

Let A_n^* be the free monoid on n ($n \geq 2$) generators.

Let R_1 and R_2 be right ideals of A_n^* .
 A morphism $\theta: R_1 \rightarrow R_2$ is a function
 such that $\theta(xa) = \theta(x)a$ ($\forall x \in R_1$)
 ($\forall a \in A_n^*$).

The polycyclic inverse monoid P_n
 is the inverse semigroup of all bijective
 morphisms between principal right ideals of A_n^* .

The Leuz equivalence relation \equiv .

Let S be a distributive inverse Λ -semigroup.

$s \equiv t$ for $s, t \in S$ iff

. $\exists 0 < \alpha \leq s$ then $t \wedge \alpha \neq 0$.

. $\exists 0 < y \leq t$ then $s \wedge y \neq 0$.

Theorem. Let S be the inverse monoid of all bijective morphisms between the f.g. right ideals of A^n .

Then S/\equiv is a Boolean inverse monoid whose group of units is isomorphic to the Thompson group G_n .

[When $n=2$, $G_2 = V$].

The above theorem can be generalized when free monoids are replaced by suitable "higher-rank graphs".

Ongoing work with Aidan Sims and

Alina Vdovina.

[Fundamental \Rightarrow group is a subgroup of the group of self-homeomorphisms of the Cantor space.]