AF inverse monoids and the structure of countable MV-algebras

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MV-algebras

An *MV-algebra* $(A, \boxplus, \neg, 0)$ is a set A equipped with a binary operation $\boxplus$, a unary operation $\neg$ and a constant 0 such that the following axioms hold.

(MV1) $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$.

(MV2) $x \boxplus y = y \boxplus x$.

(MV3) $x \boxplus 0 = x$.

(MV4) $\neg\neg x = x$.

(MV5) $x \boxplus \neg 0 = \neg 0$. Define $1 = \neg 0$.

(MV6) $\neg(\neg x \boxplus y) \boxplus y = \neg(\neg y \boxplus x) \boxplus x$.

Examples

1. Every Boolean algebra is an MV-algebra when $\vee$ is interpreted as $\boxplus$ and $\bar{\ }$ as $\neg$.

2. The real closed interval $[0,1]$ equipped with the operations $x \boxplus y = \min(1, x+y)$ and $\neg x = 1 - x$ is an MV-algebra.

3. For each $n \geq 2$ define
$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$$
equipped with the operations $\boxplus$ and $\neg$ as in (2). These are called *Łukasiewicz chains*.

4. MV-algebras arise as Lindenbaum algebras of many-valued logic in the same way that Boolean algebras arise as Lindenbaum algebras of classical, two-valued logic.

Theorems

1. The idempotents of an MV-algebra form a Boolean algebra. Thus MV-algebras are 'non-idempotent Boolean algebras'.

2. The finite MV-algebras are finite direct products of MV-algebras of the form L_n.

3. Let G be a lattice-ordered abelian group. Let $u \geq 0$ be an *order-unit* in G — thus for each $x \in G$ we have that $x \leq nu$ for some natural number n. Then $[0, u]$ is an MV-algebra where $x \boxplus y = u \wedge (x + y)$ and $\neg x = u - x$. Every MV-algebra arises in this way.

Further reading

Daniele Mundici, Logic of infinite quantum systems, *Int. J. Theor. Phys.* **32** (1993), 1941–1955.

Daniele Mundici, MV-algebras: A short tutorial, May 26, 2007.

Boolean algebras as partial algebras

In Boole's original work on Boolean algebras the operation $\boxplus$, that is $\vee$, was a partial operation defined only between orthogonal elements.

Here is an axiomatization of Boolean algebras in these terms due to Foulis and Bennett.

Let $(B, \oplus, 0, 1)$ be a set B equipped with a *partial binary operation* $\oplus$ and two constants 0 and 1 such that the following axioms hold.

(PB1) $p \oplus q$ is defined if and only if $q \oplus p$ is defined, and when both are defined they are equal.

(PB2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined then $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(PB3) For each p there is a unique q such that $p \oplus q = 1$.

(PB4) If $1 \oplus p$ is defined then $p = 0$.

(PB5) If $p \oplus q$ and $p \oplus r$ and $q \oplus r$ are defined then $(p \oplus q) \oplus r$ is defined.

(PB6) Given p and q there exist a, b, c such that $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $p = a \oplus c$ and $q = b \oplus c$.

MV-algebras as partial algebras

Let $(B, \oplus, 0, 1)$ be a set B equipped with a partial binary operation $\oplus$ and two constants 0 and 1. It is called an *effect algebra* if the following axioms hold.

(EA1) $p \oplus q$ is defined if and only if $q \oplus p$ is defined, and when both are defined they are equal.

(EA2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined then $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(EA3) For each p there is a unique p' such that $p \oplus p' = 1$.

(EA4) $1 \oplus p$ is defined if and only if $p = 0$.

Define $p \leq q$ if and only if $p \oplus r = q$ for some r.

The *refinement property* is defined as follows. If $a_1 \oplus a_2 = b_1 \oplus b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} \oplus c_{12}$ and $a_2 = c_{21} \oplus c_{22}$, and $b_1 = c_{11} \oplus c_{21}$ and $b_2 = c_{12} \oplus c_{22}$.

	b_1	b_2
a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}

Theorem *An effect algebra which is a lattice with respect to $\leq$ and satisfies the refinement property is an MV-algebra when we define*

$$a \boxplus b = a \oplus (a' \wedge b)$$

and every MV-algebra arises in this way.

Further reading

D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.*, **24** (1994), 1331–1352.

M. K. Bennett and D. J. Foulis, Phi-symmetric effect algebras, *Found. Phys.*, **25** (1995), 1699–1722.

D. J. Foulis, MV and Heyting effect algebras, *Found. Phys.*, **30** (2000), 1687–1706.

Boolean inverse monoids

An inverse monoid is said to be *Boolean* if all binary compatible joins exist, multiplication distributes over any such binary joins, and the semilattice of idempotents forms a Boolean algebra with respect to the natural partial order.

Symmetric inverse monoids are Boolean. The symmetric inverse monoid on n letters is denoted by I_n.

Boolean inverse monoids should be viewed as non-commutative generalizations of Boolean algebras.

This raises the question of how Boolean inverse monoids are related to MV-algebras.

Let S be an arbitrary Boolean inverse monoid. Put

$$\mathsf{E}(S) = E(S)/\mathscr{D}.$$

We denote the $\mathscr{D}$-class containing the idempotent e by $[e]$.

Define $[e] \oplus [f]$ as follows. If we can find idempotents $e' \in [e]$ and $f' \in [f]$ such that e' and f' are orthogonal then define $[e] \oplus [f] = [e' \vee f']$, otherwise, the operation $\oplus$ is undefined. Put $\mathbf{0} = [0]$ and $\mathbf{1} = [1]$.

An inverse monoid is *factorizable* if each element is beneath an element of the group of units.

Theorem *Let S be a Boolean inverse monoid. Then $(\mathsf{E}(S), \oplus, \mathbf{0}, \mathbf{1})$ is an effect algebra (satisfying the refinement property) if and only if S is factorizable.*

Proposition Let S be a Boolean inverse monoid.

1. S is factorizable if and only if $\mathscr{D}$ preserves complementation.

2. If S is factorizable then $\mathscr{D} = \mathscr{J}$.

3. If S is factorizable then $E(S)/\mathscr{D}$ can be replaced by $S/\mathscr{J}$.

A factorizable Boolean inverse monoid is called a *Foulis monoid*. An inverse monoid S in which $S/\mathscr{J}$ is a lattice is said to satisfy the *lattice condition*.

Theorem *Let S be a Foulis monoid satisfying the lattice condition. Then* E(S) *is an MV-algebra.*

Co-ordinatizations

We say that an MV-algebra A can be *co-ordinatized* if there is a Foulis monoid S satisfying the lattice condition such that E(S) is isomorphic to A.

Theorem 1 [Lawson, Scott, 2014] *Every countable MV-algebra can be co-ordinatized.*

Theorem 2 [Wehrung, 2015] *Every MV-algebra can be co-ordinatized.*

M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, arXiv:1408.1231v2.

F. Wehrung, Refinement monoids, equidecomposability types, and Boolean inverse semigroups, 205pp, 2015, <hal-01197354>.

Autour de Théorème 1

We can easily prove that finite MV-algebras can be co-ordinatized.

Theorem *The finite, fundamental Boolean inverse monoids are precisely the finite direct products of finite symmetric inverse monoids.*

Finite, fundamental Boolean inverse monoids are said to be *semisimple*.

Theorem *The finite MV-algebras are co-ordinatized by the semisimple monoids*.

An inverse monoid is a *meet-monoid* if all binary meets exist.

Lemma *Finite Boolean inverse monoids are meet-monoids.*

A *morphism* between Boolean inverse meet-monoids is a monoid homomorphism that maps zero to zero, preserves all compatible binary joins and all binary meets.

Proposition *A morphism between Boolean inverse meet-monoids is injective if and only if its kernel is zero.*

Proposition *Let*

$$S_0 \stackrel{\tau_0}{\to} S_1 \stackrel{\tau_1}{\to} S_2 \stackrel{\tau_2}{\to} \ldots$$

be a sequence of Boolean inverse meet-monoids and injective morphisms. Then the direct limit $\varinjlim S_i$ *is a Boolean inverse meet-monoid. In addition, we have the following.*

1. *If all the* S_i *are fundamental then* $\varinjlim S_i$ *is fundamental.*

2. *If all the* S_i *are factorizable then* $\varinjlim S_i$ *is factorizable.*

3. *The group of units of* $\varinjlim S_i$ *is the direct limit of the groups of units of the* S_i*.*

An *AF inverse monoid* is an inverse monoid isomorphic to a direct limit of semisimple monoids.

They are fundamental, factorizable Boolean inverse meet-monoids.

In particular, AF inverse monoids are Foulis monoids.

The theorem we actually proved is the following.

Theorem *Every countably infinite MV-algebra is co-ordinatized by an AF inverse monoid.*

Example The *dyadic inverse monoid* Ad_2 is the direct limit of the sequence

$$I_1 \to I_2 \to I_4 \to I_8 \to \ldots$$

Recall that a non-negative rational number is said to be *dyadic* if it can be written in the form $\frac{a}{2^b}$ for some natural numbers a and b. The dyadic rationals in the closed unit interval $[0,1]$ form an MV-algebra that is co-ordinatized by Ad_2.

Daniele Mundici, Interpretations of AF C^*-algebras in Lukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986), 15–63.

Idea of the proof

Proposition

1. *There is a morphism from I_m to I_n if and only if $m \mid n$.*

2. *If $m \mid n$ then there is exactly one morphism from I_m to I_n up to isomorphism.*

This enables us to use arguments from C^*-algebra theory in classifying morphisms between semisimple monoids. See Chapters 16 and 17 of the following.

K. R. Goodearl, *Notes on real and complex C^*-algebras*, Shiva Publishing Limited, 1982.

In particular, AF inverse monoids can be described in terms of *Bratteli diagrams*.

- Each countable MV-algebra is isomorphic to an interval $[0, u]$ in a countable lattice-ordered abelian group G.

- Countable lattice-ordered groups are dimension groups.

- Dimension groups are direct limits of groups of the form $\mathbb{Z}^r$ where the morphisms are encoded by a Bratteli diagram.

- The order-unit u arises from

$$\mathbf{n} = (n(1), \ldots, n(r)) \in \mathbf{Z}^r$$

being positive integers.

- We use $\mathbf{n}$ to construct the semisimple monoid $I_{n(1)} \times \ldots \times I_{n(r)}$ and the Bratteli diagram to encode the morphisms between the semisimple monoids.

The AF inverse monoid S that arises in this way is such that $S/\mathscr{J}$ is isomorphic to $[0, u]$.

Remarks

1. This work is further evidence of the close connectin between Boolean inverse monoids and C^*-algebras.

2. MV-algebras can be regarded as being *invariants*.

3. The two theorems suggest trying to translate theorems between Foulis monoids satisfying the lattice condition and MV-algebras. For example, is every such monoid a subdirect product of Foulis monoids in which the lattice of principal ideals is linearly ordered?