Categorical and semigroup-theoretic descriptions of Bass-Serre theory

Mark V Lawson
Heriot-Watt University
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Outline of talk

1. Motivation: the monoid case

2. Equidivisible categories

3. Skeletal Rees categories

4. The two main theorems

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1. Motivation: the monoid case

Let X be a finite alphabet and let X^* be the free monoid on X.

We define what is meant by a *self-similar group action* of the group G on X^*.

There are two maps

$$G \times X^* \to X^*,$$

denoted by $(g, x) \mapsto g \cdot x$, and

$$G \times X^* \to G,$$

denoted by $(g, x) \mapsto g|_x$, satisfying the following eight axioms:

(SS1) $1 \cdot x = x$.

(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3) $g \cdot 1 = 1$.

(SS4) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

(SS5) $g|_1 = g$.

(SS6) $g|_{xy} = (g|_x)|_y$.

(SS7) $1|_x = 1$.

(SS8) $(gh)|_x = g|_{h \cdot x} h|_x$.

All of this data may be packaged into one structure, a monoid, as follows.

On the set $X^* \times G$ define a binary operation as follows.

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

Then we get a monoid $X^* \bowtie G$, called the *Zappa-Szép product* of X^* and G.

We may exactly characterize the monoids that arise in this way.

A monoid S is said to be a *left Rees monoid* if it satisfies the following three axioms

(LR1) S is a left cancellative monoid.

(LR2) Incomparable principal right ideals are disjoint; that is, the monoid is right rigid.

(LR3) Each principal right ideal is properly contained in only a finite number of principal right ideals.

We shall usually assume that our left Rees monoids are *proper* meaning that they are not merely groups.

Theorem [Perrot 1972, Lawson 2008] *There is a correspondence between self-similar group actions and left Rees monoids.*

Let (G, X) be a self-similar group action. We say the action is *irreducible* if the action of G on X is transitive.

A left Rees monoid is said to be *irreducible* if there is a maximum proper principal ideal.

Proposition *The irreducible self-similar group actions correspond to the irreducible left Rees monoids.*

Proposition *A left Rees monoid is either irreducible or a free product with amalgamation of irreducible left Rees monoids having the same groups of units.*

A necessary condition for a monoid to be embeddable in a group is that it be cancellative.

The following motivates this whole talk.

Theorem

1. *Each irreducible Rees monoid may be embedded in its universal group.*

2. *That group is an HNN-extension over a single stable letter.*

3. *Every such group arises in this way.*

Corollary *Every Rees monoid is embeddable in a group.*

Can we generalize Rees monoids in such a way that we obtain the theory of graphs of groups as a special case?

The short answer is in the affirmative; the long answer is the contents of the rest of this talk.

2. Equidivisible

A category C is said to be *equidivisible* if for every commutative square $\begin{array}{ccc} \cdot & \xrightarrow{a} & \cdot \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ \cdot & \xrightarrow[d]{} & \cdot \end{array}$ we either have an arrow u making the following diagram commute $\begin{array}{ccc} \cdot & \xrightarrow{a} & \cdot \\ {\scriptstyle c}\big\downarrow & \overset{u}{\nearrow} & \big\downarrow{\scriptstyle b} \\ \cdot & \xrightarrow[d]{} & \cdot \end{array}$ or one, v, making the following diagram commute $\begin{array}{ccc} \cdot & \xrightarrow{a} & \cdot \\ {\scriptstyle c}\big\downarrow & \overset{v}{\swarrow} & \big\downarrow{\scriptstyle b} \\ \cdot & \xrightarrow[d]{} & \cdot \end{array}$

A non-invertible element a of a category C is called an *atom* if $a = bc$ implies either b or c is invertible. We shall always assume that there atoms.

A *length functor* is a functor $\lambda\colon C \to \mathbb{N}$ from a category C to the additive monoid of natural numbers satisfying the following conditions:

(LF1) If xy is defined then $\lambda(xy) = \lambda(x) + \lambda(y)$.

(LF2) $\lambda^{-1}(0)$ consists of all and only the invertible elements of C.

(LF3) $\lambda^{-1}(1)$ consists of all and only the atoms of C.

A *Levi category* is an equidivisible category equipped with a length functor.

A left cancellative Levi category is called a *left Rees category*.

Example Left Rees categories with one identity are precisely left Rees monoids.

Theorem *Left Rees categories are Zappa-Szép products of free categories and groupoids.*

Theorem *Free categories are precisely the Levi categories in which the invertible elements are trivial.*

Let C be a Levi category. Denote the groupoid of invertible elements by G and the set of atoms by X.

There are two groupoid actions $G \times X \to X$ and $X \times G \to X$ induced by multiplication.

The set X equipped with these actions is what we call a (G,G)-*bimodule* or simply a *bimodule*.

Remark

- If C is *left* cancellative the action $X \times G \to X$ is *(right) free*.

- If C is cancellative the action is *bifree*.

Let X be an arbitrary (G, G)-bimodule. Define

$$\mathsf{T}(X) = \bigcup_{n=0}^{\infty} X^{\otimes n}.$$

We shall call this the *tensor category* associated with the bimodule.

Theorem *With the above definition, we have the following:*

1. $\mathsf{T}(X)$ *is a Levi category and every Levi category is constructed in this way.*

2. *Left Rees categories correspond to the case where the bimodule is right free.*

3. *Rees categories correspond to the case where the bimodule is bifree.*

3. Skeletal Rees categories

A category is *skeletal* if any invertible element must belong to a local monoid.

Recall that a Rees category is a cancellative Levi category and can be constructed from a bifree bimodule.

Let G and H be groups. A *partial isomorphism* from G to H is an isomorphism $\theta: G' \to H'$ where G' is a subgroup of G and H' is a subgroup of H.

If $G = H$ we get a *partial automorphism*.

Remark HNN-extensions of a group are constructed from partial automorphisms of that group.

We shall now explain how to construct bifree bimodules from partial isomorphisms of groups.

Let D be a directed graph. An edge x from the vertex e to the vertex f will be written $e \xrightarrow{x} f$.

With each vertex e of D we associate a group G_e, called the *vertex group*, and with each edge $e \xrightarrow{x} f$, we associate a surjective homomorphism $\phi_x\colon (G_e)_x^+ \to (G_f)_x^-$ where $(G_e)_x^+ \leq G_e$ and $(G_f)_x^- \leq G_f$.

In other words, with each edge $e \xrightarrow{x} f$, we associate a partial homomorphism ϕ_x from G_e to G_f.

We call this structure a *diagram of partial homomorphisms*. If all the ϕ_x are *isomorphisms* then we shall speak of a *diagram of partial isomorphisms*.

Theorem *From each diagram of partial isomorphisms we may construct a bifree bimodule over the groupoid G which is the disjoint union of the vertex groups of the diagram.*

Groupoids which are just disjoint unions of groups are said to be *totally disconnected*.

Idea of proof

First: from partial isomorphism to bimodule

Let θ be a partial isomorphism from G to H where $\theta\colon G' \to H'$. We shall construct a set X and a (G,H)-bimodule $G \times X \times H \to X$.

On the set $G \times H$ define a relation $\equiv$ as follows: $(g_1,h_1) \equiv (g_2,h_2)$ if and only if $g_2^{-1}g_1 \in G'$ and $\theta(g_2^{-1}g_1) = h_2h_1^{-1}$.

Denote the $\equiv$-class containing (g,h) by $[g,h]$. Put $X = (G \times H)/\equiv$.

Define $g[g_1,h_1] = [gg_1,h_1]$ and $[g_1,h_1]h = [g_1,h_1h]$.

Then X is a (G,H)-biset which is bifree. In addition, $G[1,1]H = X$.

Second: from bimodule to partial isomorphism

Let X be a (G, H)-bimodule which is bifree and such that $GxH = X$. We show how to construct a partial isomorphism from G to H.

Put

$$G' = \{g \in G : gx = xh \text{ for some } h \in H\}$$

and

$$H' = \{h \in H : gx = xh \text{ for some } g \in G\}.$$

For each $g \in G'$ define

$$gx = x\theta(g).$$

Then $\theta: G' \to H'$ is an isomorphism.

4. The two main theorems

Theorem 1 *From each diagram of partial isomorphisms we may construct a skeletal Rees category, and every skeletal Rees category arises in this way.*

In particular, we may construct skeletal Rees categories from graphs of groups.

In fact, there is a direct construction of the Rees category from the diagram of isomorphisms using category presentations.

Theorem 2

1. *Every skeletal Rees category may be embedded in its universal groupoid.*

2. *When the skeletal Rees category arises from a graph of groups the universal groupoid is the fundamental groupoid of the graph of groups.*

   In addition, the Bass-Serre tree of the graph of groups arises from the way the Rees category is embedded in its universal groupoid.

5. The inverse connection and further work

We may construct inverse semigroups from skeletal Rees categories using a standard construction.

These inverse semigroups are *strongly E^*-unitary*.

The construction of the Bass-Serre tree can be achieved using the Maximum Enlargement Theorem.

It follows that the theory of graphs of groups is related to the theory of E-unitary inverse semigroups and the P-theorem.

Our theory can be viewed as a refinement of the theory developed by Ph. Higgins, in that we are replacing groupoids by ordered groupoids.