II. Boolean inverse monoids, étale groupoids and groups

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A propos

Boolean inverse semigroups are outré algebraic objects which are currently à la mode with some cachet. The theory being developed represents a rapprochement between pseudogroups of transformations and inverse semigroup theory. The key aperçu is that Boolean inverse monoids are closely related to C^*-algebras of real rank zero. Semigroups are often viewed as déclassé but if you believe in étale groupoids then you are forced by duality to believe in inverse semigroups.

Mise en scéne

The goal of this talk is to describe aspects of the theory, applications and examples of Boolean inverse semigroups.

Some of the results were inspired by the work of Hiroki Matui on étale groupoids whose calculations frequently take place in the Boolean inverse monoid associated with the étale groupoid that is his main interest.

1. Previously

Commutative	Non-commutative
frame	pseudogroup
dist. lattice	dist. inverse semigroup
Boolean algebra	Boolean inverse semigroup

Non-commutative	Groupoids
pseudogroup	étale
spatial pseudogroup	sober étale
dist. inverse semigroup	spectral étale
fundamental dist. inverse semigroup	effective spectral étale
Boolean inverse semigroups	Boolean étale

Algebra	Topology
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

Theorem [Non-commutative Stone duality]

1. *If G is a Boolean groupoid then the set of all compact-open local bisections* $\mathsf{KB}(G)$ *is a Boolean inverse semigroup; it is a monoid if the space of identities of G is compact.*

2. *If S is a Boolean inverse semigroup then the set of prime filters (= ultrafilters)* $\mathsf{G}_P(S)$ *is a Boolean groupoid; the space of identities of this groupoid is compact if S is a monoid.*

3. *S is a meet-semigroup if and only if* $\mathsf{G}_P(S)$ *is Hausdorff.*

4. *S is fundamental if and only if* $\mathsf{G}_P(S)$ *is effective.*

Boolean inverse semigroups are non-commutative generalizations of Boolean algebras.

Example Finite Boolean algebras are isomorphic to power sets of finite sets whereas finite Boolean inverse semigroups are isomorphic to all local bisections of finite discrete groupoids.

2. Simplicity

Let S be an inverse semigroup. An *ideal* I in S is a non-empty subset such that $SIS \subseteq I$.

Now let S be a Boolean inverse semigroup. A $\vee$-*ideal* I in S is an ideal with the additional property that it is closed under finite compatible joins.

A Boolean inverse semigroup is said to be 0-*simplifying* if it has no non-trivial $\vee$-ideals.

Key definition A Boolean inverse semigroup that is both fundamental and 0-simplifying is said to be *simple*.

Recall that every groupoid is a union of its connected components.

A subset of a groupoid that is a union of connected components is said to be *invariant*.

An étale groupoid is said to be *minimal* if there are no non-trivial open invariant subsets.

Theorem *Under non-commutative Stone duality,* 0*-simplifying Boolean inverse semigroups correspond to minimal Boolean groupoids*.

A non-zero idempotent e is said to be *properly infinite* if we may find a pair of elements $x, y \in S$ such that $e \xrightarrow{x} i < e$ and $e \xrightarrow{y} j < e$ and $i \perp j$.

An inverse monoid is said to be *purely infinite* if every non-zero idempotent is properly infinite.

Theorem *For an atomless Boolean inverse semigroup S the following are equivalent.*

1. *S is* 0*-simple.*

2. *S is* 0*-simplifying and purely infinite.*

An inverse semigroup is said to be 0-*simple* if it has no non-trivial ideals.

Thus 0-simple is stronger than 0-simplifying.

The following is a version of a classical theorem.

Theorem *A Boolean inverse semigroup has no non-trivial (semigroup) congruences if and only if it is* 0*-simple and fundamental.*

Definition A Boolean inverse semigroup that is both fundamental and 0-simple is said to be *strongly simple*.

A non-zero element a in an inverse semigroup is said to be an *infinitesimal* if $a^2 = 0$. The following result explains why infinitesimals are important.

Proposition *Let S be a Boolean inverse monoid and let a be an infinitesimal. Then*

$$a \vee a^{-1} \vee \overline{(a^{-1}a \vee aa^{-1})}$$

is an involution.

We call an involution that arises in this way a *transposition*.

A Boolean inverse semigroup is said to be *basic* if each element is a finite join of infinitesimals or idempotents. A groupoid is *principal* if it comes from an equivalence relation.

Theorem *Under non-commutative Stone duality basic Boolean inverse semigroups correspond to principal Boolean groupoids.*

Summary

Inverse monoid	**Etale groupoid**
fundamental	effective
0-simplifying	minimal
0-simple	purely infinite and minimal
basic	principal

3. Groups, inverse semigroups and groupoids

Idea The groups of units of simple Boolean inverse monoids should be regarded as generalizations of finite symmetric groups.

Theorem [The simple alternative] *A simple Boolean inverse monoid is either isomorphic to a finite symmetric inverse monoid or atomless.*

Under classical Stone duality, the Cantor space corresponds to the (unique) countable atomless Boolean algebra; it is convenient to give this a name and we shall refer to it as the *Tarski* algebra.

Corollary *A simple countable Boolean inverse monoid has the Tarski algebra as its set of idempotents.*

Definition Denote by Homeo($\mathscr{S}$) the group of homeomorphisms of the Boolean space $\mathscr{S}$. By a *Boolean full group*, we mean a subgroup G of Homeo($\mathscr{S}$) satisfying the following condition: let $\{e_1, \ldots, e_n\}$ be a finite partition of $\mathscr{S}$ by clopen sets and let $g_1, \ldots, g_n$ be a finite subset of G such that $\{g_1 e_1, \ldots, g_n e_n\}$ is a partition of $\mathscr{S}$ also by clopen sets. Then the union of the partial bijections $(g_1 \mid e_1), \ldots, (g_n \mid e_n)$ is an element of G. We call this property *fullness* and term *full* those subgroups of Homeo($\mathscr{S}$) that satisfy this property.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups.*

2. *Simple Boolean inverse monoids*

3. *Minimal, effective Boolean groupoids.*

Let $\mathscr{S}$ be a compact Hausdorff space. If $\alpha \in$ Homeo($\mathscr{S}$), define

$$\text{supp}(\alpha) = \text{cl}\{x \in \mathscr{S} \colon \alpha(x) \neq x\}$$

the *support* of α.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups in which each element has clopen support.*

2. *Simple Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff Boolean groupoids.*

Theorem *The following three classes of structure are equivalent.*

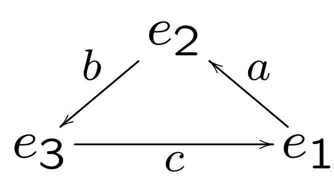
1. *Minimal Boolean full groups in which each element has a* clopen fixed-point set*.*

2. *Simple basic Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff, principal Boolean groupoids.*

Let S be a Boolean inverse monoid. Denote by Sym(S) the subgroup of the group of units of S generated by transpositions.

Let a and b be infinitesimals and put $c = (ba)^{-1}$ as in the following diagram



where the idempotents e_1, e_2, e_3 are mutually orthogonal. Put $e = e_1 \vee e_2 \vee e_3$. Then $a \vee b \vee c \vee \bar{e}$ is a unit called a 3-*cycle*.

Denote by Alt(S) the subgroup of the group of units of S generated by 3-cycles.

Theorem [Nekrashevych] *Let S be a simple Boolean inverse monoid. Then* Alt(S) *is simple.*

4. Examples

The first example was in fact the first one I constructed and motivated the whole theory. The starting point was a paper by J.-C. Birget in 2004.

Example 1

There is a family $C_2, C_3, \ldots$ of strongly simple, countable atomless Boolean inverse monoids, the *Cuntz inverse monoids*, whose groups of units are the Thompson groups $V_2, V_3, \ldots$, respectively.

The groupoid associated with C_n is the same as the groupoid associated with the Cuntz C^*-algebra $\mathcal{O}_n$.

Representations of the inverse monoids C_n are (unknowingly) the subject of *Iterated function systems and permutation representations of the Cuntz algebra* by O. Bratteli and P. E. T. Jorgensen, AMS, 1999.

Example 2

The *AF monoids* are a class of fundamental Boolean inverse monoids defined to be direct limits $\varinjlim S_i$ where the inverse semigroups S_i are finite direct products of finite symmetric inverse monoids and the maps between them preserve joins.

Their groups of units are direct limits of finite direct products of finite symmetric groups with morphisms being by means of diagonal embeddings.

AF monoids can be used to co-ordinatize MV algebras.

End of lecture. The following material is additional.

5. Constructing Boolean inverse semigroups

The first theorem tells us that if we are interested in inverse semigroups embedded (nicely) in C^*-algebras then we might as well assume that the inverse semigroup is Boolean.

Theorem [Paterson, Wehrung] *Let S be an inverse subsemigroup of the multiplicative semigroup of a C^*-algebra R in such as way that the inverse in S is the involution of R. Then there is a Boolean inverse semigroup B such that $S \subseteq B \subseteq R$ such that the inverse in B is the involution in R.*

There are two ways of constructing a Boolean inverse semigroup from a (distributive) inverse semigroup: off-the-peg (which is easy) and bespoke (which is more delicate).

To describe the off-the-peg method, it is convenient to restrict to distributive inverse semigroups. This is actually no restriction by the following result.

Theorem *Let S be an inverse semigroup with zero. Then there is a distributive inverse semigroup* $\mathsf{D}(S)$ *and an embedding* $\delta\colon S \to \mathsf{D}(S)$ *universal for maps from S to distributive inverse semigroups.*

The following generalizes Grätzer and Schmidt (1958) by way of results by Kellendonk, Paterson, Lenz, and Lawson, Margolis & Steinberg.

Theorem [The enveloping Boolean inverse semigroup] *Let S be a distributive inverse semigroup. Then there is a Boolean inverse semigroup* BS(S) *and an embedding $\beta\colon S \to$* BS(S) *such that each element of* BS(S) *can be written in the form*

$$\bigvee_{i=1}^{m} \beta(s_i) \setminus \beta(t_i)$$

where $t_i \leq s_i$.

The bespoke method is part of the theory of *coverages on inverse semigroups*.

The most important such coverage is the tight coverage. The term 'tight' come from the work of Ruy Exel but there were related notion in work by Lenz and Lawson.

Let S be an inverse semigroup. Let $a \in S$. A *tight cover* of a is a finite subset $X \subseteq a^{\downarrow}$ such that for each $0 \neq b \leq a$ there exists $x \in X$ such that there is a $0 \neq z$ where $z \leq b, x$.

For each $a \in S$ define $\mathcal{T}(a)$ to be the set of tight covers of a.

Let D be a distributive inverse semigroup. A morphism $\theta \colon S \to D$ is said to be a *cover-to-join* map if $X \in \mathcal{T}(a)$ implies that $a = \bigvee \theta(X)$.

Theorem *Let S be an inverse semigroup. Then there is a universal cover-to-join map $\delta_t \colon S \to \mathsf{D}_t(S)$.*

A filter A is said to be *tight* if $a \in A$ and $X \in \mathcal{T}(a)$ implies that $X \cap A \neq \emptyset$.

The following is a version of a theorem by Exel.

Theorem *The distributive inverse semigroup $\mathsf{D}_t(S)$ is Boolean if and only if every tight filter in S is an ultrafilter.*

The étale groupoid associated with $\mathsf{BS}(\mathsf{D}_t(S))$ is Exel's *tight groupoid*.

6. Research directions

- Fred Wehrung has shown that Boolean inverse semigroups form a variety. What do the free Boolean inverse semigroups look like? How does the theory of subvarieties of simple Boolean inverse monoids mesh with the behavio(u)r of their groups of units?

- Matui's work has highlighted the importance of the (integer) homology groups of Boolean étale groupoids. But in fact many of his calculations dealing with these groups take place within the associated Boolean inverse monoid. Investigate the connection between the structure of the Boolean inverse monoid and the structure of the homology groups.

- Peter Hines investigated the connections between inverse semigroups and linear logic (‘geometry of interaction’). Does the theory of Boolean inverse monoids shed any light on these connections? (there are hints that they might do.)

- A *Tarski inverse monoid* is a countable, atomless Boolean inverse monoid. ‘Classify’ the simple Tarski inverse monoids.

- Formalize the connection netween Boolean inverse monoids and C^*-algebras of real rank zero.

- What can be said about the Boolean inverse monoids associated with aperiodic tilings à la Kellendonk and, specifically, their groups of units?