I. An introduction to Boolean inverse semigroups

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0. In principio

The monograph

> J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer, 1980.

identified connections between inverse semigroups, étale groupoids and C^*-algebras.

This talk is about the nature of the connection between inverse semigroups and étale groupoids.

1. Pseudogroups of transformations

Let X be a topological space. A *pseudogroup of transformations on* X is a collection Γ of homeomorphisms between the open subsets of X (called *partial homeomorphisms*) such that

1. Γ is closed under composition.

2. Γ is closed under 'inverses'.

3. Γ contains all the identity functions on the open subsets.

4. Γ is closed under arbitrary non-empty unions when those unions are partial bijections.

Example Let X be endowed with the discrete topology. Then the set $I(X)$ of all partial bijections on X is a pseudogroup.

Observations on pseudogroups

- Pseudogroups important in the foundations of geometry.

- The idempotents in Γ are precisely the identity functions on the open subsets of the topological space. They form a complete, infinitely distributive lattice or *frame*.

- Johnstone on the origins of frame theory

> It was Ehresmann ... and his student Bénabou ... who first took the decisive step in regarding complete Heyting algebras as 'generalized topological spaces'.

However, Johnstone does not say *why* Ehresmann was led to his frame-theoretic viewpoint of topological spaces. The reason was pseudogroups.

- Pseudogroups usually replaced by their *groupoids of germs* but pseudogroups nevertheless persist.

- The algebraic part of pseudogroup theory became inverse semigroup theory.

But recent developments show that it is fruitful to bring these divergent approaches back together.

In particular, inverse semigroups and frames.

This is very much in the spirit of Ehresmann's work.

2. Inverse semigroups

Inverse semigroups arose by abstracting pseudogroups of transformations in the same way that groups arose by abstracting groups of transformations.

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Wagner (1908–1981) in the USSR.

They all three converge on the definition of 'inverse semigroup'.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

The idempotents in an inverse semigroup commute with each other. We speak of the *semilattice of idempotents* $E(S)$ of the inverse semigroup S.

Pseudogroups of transformations are inverse semigroups.

The pseudogroups $I(X)$ are called *symmetric inverse monoids*.

Theorem [Wagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

The definition of pseudogroups requires unions and so order.

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup.*

It is called the *natural partial order*.

Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set is compatible.

In the symmetric inverse monoid $I(X)$ the natural partial order is defined by restriction of partial bijections.

The union of two partial bijections is a partial bijection if and only if they are compatible.

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

Pseudogroups are the correct abstractions of pseudogroups of transformations.

But this leads us to think of inverse semigroup theory from a lattice-theoretic perspective.

An inverse semigroup is a *meet-semigroup* if has has all binary meets.

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Summary

Commutative	**Non-commutative**
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet-semigroup

3. Etale groupoids

The classical theory of pseudogroups of transformations requires topology.

We generalize the classical connection between topological spaces and frames.

To each topological space X there is the associated frame of open sets $\Omega(X)$.

To each frame L there is the associated topological space of completely prime filters $\mathsf{Sp}(L)$.

The following is classical.

Theorem *The functor* $L \mapsto \mathit{Sp}(L)$ *from the dual of the category of frames to the category of spaces is right adjoint to the functor* $X \mapsto \Omega(X)$.

A frame is called *spatial* if elements can be distinguished by means of completely prime filters.

A space is called *sober* if points and completely prime filters are in bijective correspondence.

But we need ‘non-commutative topological spaces’.

Topological groupoids will be just the ticket.

We view categories as 1-sorted structures (over sets): everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as ‘groups with many identities’.

Key definition Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet-monoid.*

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

Etale groupoids therefore have a strong algebraic character.

There are two basic constructions.

- Let G be an étale groupoid. Denote by $\mathsf{B}(G)$ the set of all open local bisections of G. Then $\mathsf{B}(G)$ is a pseudogroup.

- Let S be a pseudogroup. Denote by $\mathsf{G}(S)$ the set of all *completely prime filters* of S. Then $\mathsf{G}(S)$ is an étale groupoid. [This is the 'hard' direction].

Denote by $\mathbf{Inv}$ a suitable category of pseudogroups and by **Etale** a suitable category of étale groupoids.

Theorem [The main adjunction] *The functor* $\mathsf{G}\colon \mathbf{Inv}^{op} \to \mathbf{Etale}$ *is right adjoint to the functor* $\mathsf{B}\colon \mathbf{Etale} \to \mathbf{Inv}^{op}$.

Theorem [The main equivalence] *There is a dual equivalence between the category of spatial pseudogroups and the category of sober étale groupoids.*

4. Non-commutative Stone dualities

An étale groupoid is said to be *spectral* if its identity space is sober, has a basis of compact-open sets and if the intersection of any two such compact-open sets is compact-open. We refer to *spectral groupoids* rather than *spectral étale groupoids*.

Theorem *There is a dual equivalence between the category of distributive inverse semigroups and the category of spectral groupoids.*

- Under this duality, a spectral groupoid G is mapped to the set of all *compact-open local bisections* $\mathsf{KB}(G)$

- Under this duality, a distributive inverse semigroup is mapped to the set of all *prime filters* $\mathsf{G}_P(S)$.

An étale groupoid is said to be *Boolean* if its identity space is Hausdorff, locally compact and has a basis of clopen sets. We refer to *Boolean groupoids* rather than *Boolean étale groupoids*.

Proposition *A distributive inverse semigroup is Boolean if and only if prime filters and ultrafilters are the same.*

Theorem *There is a dual equivalence between the category of Boolean inverse semigroups and the category of Boolean groupoids.*

Theorem *There is a dual equivalence between the category of Boolean inverse* meet-semigroups *and the category of* Haudorff *Boolean groupoids.*

The theorems stated above are for semigroups (and so refer to *non-unital* distributive lattices and Boolean algebras). In the monoid cases, the spaces of identities become compact.

Algebra	**Topology**
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

Papers

P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

M. V. Lawson, D.H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.

Let G be an étale groupoid.

Its *isotropy subgroupoid* $\mathrm{Iso}(G)$ is the subgroupoid consisting of the union of its local groups.

The groupoid G is said to be *effective* if the interior of $\mathrm{Iso}(G)$, denoted by $\mathrm{Iso}(G)^\circ$, is equal to the space of identities of G.

Let S be an inverse semigroup. It is *fundamental* if the only elements that commute with all idempotents are idempotents.

The following is classical.

Theorem *Every inverse semigroup has a fundamental image by means of a homomorphism that is injective when restricted to the semilattice of idempotents.*

Theorem *Under the dual equivalences.*

1. *Fundamental spatial pseudogroups correspond to effective sober étale groupoids.*

2. *Fundamental distributive inverse semigroups correspond to effective spectral groupoids.*

3. *Fundamental Boolean inverse semigroups correspond to effective Boolean groupoids.*

5. Boolean inverse semigroups

These are currently the most interesting class of inverse semigroups.

Theorem [Classification of finite Boolean inverse semigroups]

1. *Each finite Boolean inverse semigroup is isomorphic to the set of all local bisections of a finite discrete groupoid.*

2. *Each finite fundamental Boolean inverse semigroup is isomorphic to a finite direct product of finite symmetric inverse monoids.*

Result (1) above should be compared with the structure of finite Boolean algebras, and result (2) with the structure of finite dimensional C^*-algebras.

There are Boolean inverse monoid analogues of

- AF C^*-algebras.
- Cuntz C^*-algebras.
- Cuntz-Krieger C^*-algebras.

These C^*-algebras all have real rank zero which raises an obvious question.

In my next talk, I shall focus on *simple* Boolean inverse semigroups, where it is possible to prove interesting theorems.

To be continued . . .