Non-commutative Stone dualities tight completions and groups

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With the collaboration of

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and thanks to

Gracinda Gomes (Lisbon) and Pedro Resende (Lisbon)

Monoids vs. semigroups

In these talks, the distinction between these two structures will be important.

Monoids correspond to *compact* spaces

Semigroups correspond to *locally compact* spaces

We shall focus on monoids in these lectures.

A *unit* in a monoid is an invertible element.

Introduction

(a) Doing group theory by mistake

In the 90's, Peter Hines and I carried out work on the mathematics underlying Girard's *geometry of interaction* programme (parts I, II and III) for linear logic.

A strange group emerged that we were unable to identify. Discussed in Section 9.3 of my inverse semigroup book.

Later, formulated a description of arbitrary inverse semigroups in terms of (equivalence classes of) ordered pairs.

Claas Röver saw a connection with Dehornoy's work and Thompson groups.

Became interested in the connection between inverse semigroups and Thompson groups.

Read: J.-C. Birget, The groups of Richard Thompson and complexity, *IJAC* **14** (2004), 569–626.

It seemed to me that a more intrinsic approach was possible (and is).

This led to:

M. V Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Communications in algebra* (2007) **35**, 4068–4087

And that appeared to be that.

(b) Inverse semigroups in C^*-algebras

Inverse semigroups were muscling in to the theory of C^*-algebras.

J. Renault, *A groupoid approach to C^*-algebras*, LNM 793, 1980.

Ostensibly about topological groupoids and C^*-algebras, but inverse semigroups keep popping up (why?).

> "Together with the notion of groupoid, the notion of inverse semi-group (sic) plays an important role in this work."

A. Kumjian, On localizations and simple C^*-algebras, *Pacific J. Math.* **112** (1984), 141–192.

> "The convenient fiction indulged in by most practitioners in the field is that C^*-algebras are somehow to be conceived of as continuous functions on a "non-commutative" topological space."

> "A localization may profitably be viewed as a non-commutative analog of a countable basis; its affiliated inverse semigroup is to be viewed as the analog of a topology."

A. L. T. Paterson *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1998.

Inverse semigroups get a mention at last in the title.

J. Kellendonk, The local structure of tilings and their integer group of coinvariants, *Commun. Math. Phys.* **187** (1997), 115–157.

J. Kellendonk, Topological equivalence of tilings, *J. Math. Phys.* **38** (1997), 1823–1842.

Interested in the physical properties *quasi-crystals*.

These are modelled by aperiodic tilings.

Associated with each such tiling is an algebraic structure called an *almost groupoid*.

Turns out, these are just inverse semigroups wearing false beards.

D. H. Lenz, On an order-based construction of a topological groupoid from an inverse semigroup, *PEMS* **51** (2008), 387–406.

[In circulation in *samizdat* form since 2002.]

Begins to join the dots — Renault, Paterson, Kellendonk.

Inverse semigroups are *vaguely* related to topological groupoids (classical result).

But how, exactly?

Epiphany/Serendipity

The work

> on constructing the Thompson groups from inverse semigroups (a)

and

> the relation between inverse semigroups and topological groupoids (b)

are *two sides of one coin*.

The argument

The relationship between inverse semigroups and topological groupoids is a consequence of a non-commutative generalization of Stone duality.

We focus in these lectures particularly on the non-commutative versions of the classical duality connecting unital Boolean algebras with a class of topological spaces.

We replace the lattices by suitable classes of semigroups (here, inverse monoids) and the spaces by suitable topological/localic categories (here, groupoids).

Groups emerge as groups of units of the participating inverse monoids.

Thus our work suggests a broader framework for studying groups of this type.

Main references

- A perspective on non-commutative frame theory, with G. Kudryavtseva, arXiv:1404.6516.

- Distributive inverse semigroups and non-commutative Stone dualities, with D. Lenz, arXiv:1302.3032.

- Pseudogroups and their étale groupoids, with D. Lenz, *Adv. Maths* **244** (2013), 117–170.

- Non-commutative Stone duality: inverse semigroups, topological groupoids and C*-algebras, *IJAC* **22**, 1250058 (2012), 47pp.

- A non-commutative generalization of Stone duality, *J. Austral. Math. Soc.* (2010) **88**, 385–404.

- The étale groupoid of an inverse semigroup as a groupoid of filters, with S. W. Margolis and B. Steinberg, *J. Austral. Math. Soc.* **94** (2014), 234–256.

and Pedro Resende's

Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

Pedro's work and ours is largely complementary, but is unified in our most recent paper.

Contents

I. Basic definitions and motivating examples

1. Primer on inverse semigroups

2. The polycyclic inverse monoids

3. Strong representations of the polycyclic inverse monoids

4. The construction of the Cuntz inverse monoids

1. Primer on inverse semigroups

I shall begin with a concrete class of examples that will, in fact, motivate *everything* I have to say.

Let X be a non-empty set. A *partial bijection* is a bijection $f: A \to B$ where $A, B \subseteq X$.

Denote by f^{-1} the partial bijection $f^{-1}: B \to A$.

Let $g: C \to D$. Suppose that $C \subseteq A$ and $D \subseteq B$ and $g(x) = f(x)$ for all $x \in C$. Then we write $g \subseteq f$. This is the *restriction ordering*.

Denote by $I(X)$ the set of all partial bijections on X.

This includes the empty partial function, denoted by 0, and the identity function defined on X, denoted by 1.

More generally, *partial identities* 1_A for $A \subseteq X$.

We now make some observations about the properties of $I(X)$ with respect to this composition.

- This composition is associative and 1 is its identity. Thus $I(X)$ is a monoid.

- $f = ff^{-1}f$ and $f^{-1} = f^{-1}ff^{-1}$, and if $g \in I(X)$ such that $f = fgf$ and $g = gfg$ then $g = f^{-1}$.

- The partial identities are idempotents and they are the *only* idempotents.

- $f^{-1}f = 1_{\mathsf{dom}(f)}$ and $ff^{-1} = 1_{\mathsf{im}(f)}$.

- $g \subseteq f$ if, and only if, $g = fg^{-1}g$.

- $1_A \subseteq 1_B$ if, and only if, $A \subseteq B$.

- $f \cup g \in I(X)$ if, and only if, $f^{-1}g$ and fg^{-1} are both idempotents.

All of this serves to motivate our next definition.

Definition and examples

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

Example The semigroups $I(X)$ are therefore inverse monoids. They are called *symmetric inverse monoids*.

Observe that $s^{-1}s$ and ss^{-1} are idempotents, and that $(s^{-1})^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1}$.

We call $s^{-1}s$ the *domain idempotent* and ss^{-1} the *range idempotent*.

It can be proved that idempotents commute (Liber/Munn and Penrose).

Set of idempotents of S, denoted by $E(S)$, equipped with an order $e \leq f$ iff $e = ef = fe$ which makes $E(S)$ a *meet-semilattice*.

We now have a theorem à la Cayley that tells us that our definition does what it should do.

Wagner-Preston representation theorem Every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse monoid.

Take home message: Inverse semigroups are to partial symmetries as groups are to (global) symmetries.

An inverse semigroup S is equipped with three important relations:

- $s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

- $s \sim t$ if, and only if, st^{-1} and $s^{-1}t$ both idempotents. *Compatibility relation*. Not in general an equivalence relation. If $a, b \leq c$ then $a \sim b$. Thus this relation controls when pairs of elements are *eligible* to have a join. A subset is *compatible* if every pair of elements in the subset are compatible.

- $s \perp t$ if, and only if, $s^{-1}t = 0 = st^{-1}$. This is the *orthogonality relation*.

Examples

1. *Groups*. A group is an inverse semigroup with exactly one idempotent and so is *degenerate*.

2. *Meet-semilattices*. These are the inverse semigroups in which every element is an idempotent.

3. *Presheaves of groups over meet-semilattices*. These are the inverse semigroup whose idempotents are central.

4. *Pseudogroups of transformations*. The inverse semigroups of all homeomorphisms of a topological space. A *pseudogroup* is an inverse monoid in which every non-empty compatible subset has a join and multiplication distributes over any joins that exist.

Two, more sophisticated, examples.

1. *Self-similar group actions*. These may be encoded by suitable inverse monoids using Zappa-Szép products. See: A correspondence between a class of monoids and self-similar group actions II, with A. Wallis, arXiv:1308.2802 *and* A correspondence between a class of monoids and self-similar group actions I, *Semigroup Forum* (2008) **76**, 489-517. Based on the 1972 thesis of J.-F. Perrot.

2. *Graphs of groups*. These may be encoded by suitable inverse semigroups since such graphs show how to glue groups together using *partial isomorphisms*. See: A categorical description of Bass-Serre theory, with A. Wallis, arXiv:1304.6854.

Misc. points

The inverse semigroups we study will almost all have a zero and this will be assumed from now on.

There are no easy analogues of normal subgroups and so morphisms between inverse semigroups are usually studied using *congruences* generalizing elementary number theory.

Ideals may be defined in the obvious way and are associated with some homomorphisms.

Inverse semigroups without any non-trivial ideals are called 0-*simple*.

Inverse semigroups without any non-trivial congruences are called *congruence-free*.

Origins in the 1950's

- *In the West.* Gordon Preston who developed ideas of David Rees (both at Bletchley Park).

- *In the East.* Viktor Vladimirovich Wagner a differential geometer interested in pseudogroups.

- *In France.* Charles Ehresmann a differential geometer interested in pseudogroups.

. . . and reinvented many times subsequently.

Two recent-ish examples are: *almost groupoids* and *combinatorial pseudogroups*.

A little more on symmetric inverse monoids

The inverse monoid $I(X)$ has extra structure.

- The semilattice of idempotents forms a Boolean algebra.
- Pairs of compatible elements have joins.
- Multiplication distributes over any joins that exist.
- All pairs of elements have meets.

Important definitions

An inverse monoid is said to be *distributive* if it has all binary compatible joins and multiplication distributes over those joins.

An inverse monoid is said to be *Boolean* if it is distributive and the semilattice of idempotents is a Boolean algebra.

An inverse monoid is a $\wedge$-*monoid* if every pair of elements has a meet.

Thus, symmetric inverse monoids are Boolean inverse $\wedge$-monoids.

- Their groups of units are interesting.

- *In the finite case*, every partial bijection may be extended to a bijection.

The inverse monoids we shall ultimately study will be natural generalizations of finite symmetric inverse monoids.

2. The polycyclic inverse monoids

After the symmetric inverse monoids, the polycyclic inverse monoids are the most important class of examples. They will also lead us to the Thompson groups.

The first example of a partial bijection we meet is the map $s: \mathbb{N} \to \mathbb{N}$ given by $n \mapsto n + 1$.

More generally, partial bijections are used to define *Dedekind infiniteness*.

The inverse submonoid of $I(\mathbb{N})$ generated by s is called the *bicyclic inverse monoid*. As it happens, this is sans zero.

We shall define a class of inverse monoids that generalizes this example. They will be called *poly*cyclic inverse monoids.

[So, nothing to do with polycyclic groups]

Let X be an infinite set. Let $X_1, \ldots, X_n \subseteq X$ be $n \geq 2$ infinite subsets, pairwise disjoint, and each having the same cardinality as X. Let $f_i: X \to X_i$ be chosen partial bijections. The inverse submonoid of $I(X)$ generated by the set $\{f_1, \ldots, f_n\}$ is called the *polycyclic monoid on* n *generators* P_n.

Remark The *the* can be justified.

Remark We do not require the set $\{X_1, \ldots, X_n\}$ to form a partition of X. BUT thereby hangs a tail (spoiler alert).

Introduced by
M. Nivat, J.-F. Perrot, Une généralisation du monoïde bicyclique, *C. R. Acad. Sci. Paris Sér. A* **271** (1971), 824–827.

The polycyclic inverse monoids are the most interesting monoids you have never heard of

- They are congruence-free.

- They are the syntactic monoids of Dyck languages.

- They may be used to recognize arbitrary context-free languages.

- They arise in the foundations of amenability. See: T. Ceccherini-Silberstein, R. Grigorchuk, P. de la Harpe, Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces, *Proc. Steklov Inst. Math.* **224** (1999), 57–97.

- They arise as structure monoids à la Dehornoy. See: M. V. Lawson, A correspondence between balanced varieties and inverse monoids, *IJAC* (2006) **16**, 887–924.

- They arise in Girard's work on linear logic (bon chance). See: PhD thesis of Peter Hines.

- They arise in the construction of the Cuntz C^*-algebras. See: J. Cuntz, Simple C^*-algebras generated by isometries, *Commun. Math. Phys.* **57** (1977), 173–185.

- They arose in the 1950's in the theory of Leavitt path algebras in the study of rings R where $R^n \cong R$ as left R-modules.

- Their representation theory (**see below**) is important in the theory of wavelets. See: O. Brattelli, P. E. T. Jorgensen, *Iterated function systems and permutations representations of the Cuntz algebra*, Memoirs A. M. S. No. 663 (1999). See also the work of Katsunori Kawamura.

The pop-push representation

Remark From now on I shall restrict to the case $n = 2$ since it will be clear how to generalize to arbitrary n.

First, some string theory.

The *free monoid* on $\{a, b\}$ is denoted by $(a+b)^*$ and consists of *strings* (or *words*, if you will). Multiplication is *concatenation* and identity is the empty string ε.

The polycyclic monoid P_2 can be regarded as consisting of a zero and all symbols of the form yx^{-1} where x and y are elements of $(a + b)^*$.

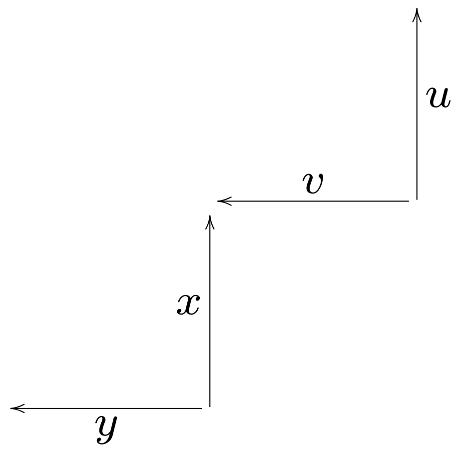
The product of two symbols yx^{-1} and vu^{-1} is zero unless x and v are *prefix-comparable*, meaning one is a prefix of the other, in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some } z \end{cases}$$

The elements of P_2 can be thought of as a combination of two operations on a pushdown stack: xy^{-1} means *pop* y *and push* x.

Notation warning!

The notation xy^{-1} is open to misinterpretation. Read y^{-1} simply as *erase* y. The above product should be visualized thus



The inverse of yx^{-1} is xy^{-1}.

The non-zero idempotents of P_2 are the elements of the form xx^{-1}.

Observe that $yy^{-1} \leq xx^{-1}$ if, and only if, x is a prefix of y.

So, longer strings are smaller.

3. Strong representations of the polycyclic monoids

A *representation* of an inverse semigroup S is a morphism to $I(X)$.

We now look at a special class of representations of P_n.

A representation θ of P_2 is *strong* if

$$1 = \theta(aa^{-1}) \vee \theta(bb^{-1}).$$

Paraphrase: If we return to our original definition of P_n, we actually require the images of our partial bijections to form a partition.

Examples

1. The representation of P_2 in $I((a+b)^*)$ is *not* strong because

$$(a+b)^* = \varepsilon + a(a+b)^* + b(a+b)^*.$$

   So, as in the construction of IKEA furniture, we have a bit left over.

2. BUT if we represent P_2 in $I((a+b)^\omega)$, where $(a+b)^\omega$ is the set of right-infinite strings, then we *do* get a strong representation since

$$(a+b)^\omega = a(a+b)^\omega + b(a+b)^\omega.$$

The theory of strong representations of the polycyclic monoids turns out to be complex and interesting.

See: Bratteli and Jorgensen (above).

See: M. V Lawson, D. Jones, Strong representation of the polycyclic inverse monoids: cycles and atoms, *Periodica Math. Hung.* (2012) **64**, 53-87.

See: PhD thesis of David G. Jones.

Related to our original work in linear logic.

See: PhD thesis of Peter Hines.

Examples Jones investigated the orbit structure of strong representations of P_2 determined by pairs of maps $\sigma_0, \sigma_p\colon \mathbb{Z} \to \mathbb{Z}$ given by $\sigma_0(n) = 2n$ and $\sigma_p(n) = 2n + p$, where p is a fixed odd number.

The maximum number of orbits occurs when $p = 2^q - 1$, a Mersenne number. The orbit structure is determined by the *binary representations* of the fractions $\frac{a}{p}$.

In the plane, Bratteli and Jorgensen show how to get *fractal tilings*.

4. The construction of the Cuntz inverse monoids

I shall now procede informally but, if you are so minded, you may see the algebraic scaffolding in:

M. V Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Comm. Alg.* (2007) **35**, 4068–4087.

As mentioned earlier, the motivation for this paper was

J.-C. Birget, The groups of Richard Thompson and complexity, *IJAC* **14** (2004), 569–626.

Birget works with semigroups rings but a more intrinsic approach is possible.

The idea

I shall glue finite sets of compatible elements of P_2 together subject to the relation

$$1 = aa^{-1} \vee bb^{-1}.$$

I am not simply trying to build units.

Example Consider the elements

$$a^2a^{-1}, \quad ab(ba)^{-1}, \quad bb^{-2}.$$

They are pairwise orthogonal. We may form the join

$$\alpha = a^2a^{-1} \vee ab(ba)^{-1} \vee bb^{-2}.$$

which has the domain idempotent

$$\alpha^{-1}\alpha = aa^{-1} \vee ba(ba)^{-1} \vee b^2b^{-2}.$$

But we may now apply our relation above to get

$$aa^{-1} \vee ba(ba)^{-1} \vee b^2b^{-2}$$

$$=$$

$$aa^{-1} \vee b(aa^{-1} \vee bb^{-1})b^{-1}$$

$$=$$

$$aa^{-1} \vee bb^{-1} = 1.$$

Easy to check that $\alpha\alpha^{-1} = 1$ and so α is a unit.

Define C_2 to consist of **all** — not just those that determine units — finite (orthogonal) joins of elements of P_2 subject to the relation

$$1 = aa^{-1} \vee bb^{-1}.$$

Now generalize to C_n.

Theorem The inverse monoid C_n has the following properties.

1. It is a Boolean inverse $\wedge$-monoid whose semilattice of idempotents is a countable atomless Boolean algebra.

2. It is congruence-free.

3. Its group of units is the Thompson group $G_{n,1}$ (or, what you will).

4. There is an embedding $P_n \to C_n$ with the property that every strong representation of P_n extends to a representation of C_n.

We call C_n the *Cuntz inverse monoid (of degree n)*.

The main theme of these lectures

We constructed the inverse monoid C_n by glueing together compatible elements of P_n subject to a relation.

The inverse semigroup P_n has a trivial group of units.

Units in C_n arise by glueing together non-units in P_n.

This leads to the following idea:

Construct groups by glueing together elements of an inverse semigroup subject to certain relations.

II. Dualities and completions

1. Classical Stone duality

2. Etale groupoids

3. Boolean inverse $\wedge$-monoids and their étale groupoids

4. Generalizations

5. How to construct Boolean inverse $\wedge$-monoids

6. Back to Cuntz inverse monoids

1. Classical Stone duality

If X is a topological space, denote the lattice of open subsets by $\Omega(X)$.

I shall restrict attention to *unital* Boolean algebras and *unital* distributive lattices, though this is not essential.

A *sober* space is (essentially) one in which open sets determine points.

A *spectral* space is a compact topological space that is sober and has a basis of compact-open sets closed under binary intersections.

A *Stone* space is a Hausdorff spectral space. Equivalently, it is a compact Hausdorff space with a basis of clopen subsets.

Stone Duality For suitable morphisms.

1. The category of unital distributive lattices is dually equivalent to the category of spectral spaces.

2. The category of unital Boolean algebras is dually equivalent to the category of Stone spaces.

I shall briefly describe the classical proof of (2) above, since this is the most relevant to these lectures.

The big idea

In kindergarten, we learn that every finite Boolean algebra is isomorphic to the powerset P(X) for some finite set X.

The proof relies on the existence of atoms.

An *atom* in a poset is an element x such that $y \leq x$ implies either $y = 0$ or $y = x$.

Finite Boolean algebras always have atoms, but infinite ones don't have to.

What to do?

Replace atoms by non-localized things called ultrafilters — a sort of quantum revolution.

Do *they* always exist?

Yes! By the magic of the Axiom of Choice.

Ultrafilters

Let X be a meet-semilattice with zero. An *ultrafilter* in X is a non-empty subset $F \subseteq X$ such that

1. $0 \notin F$.

2. $x, y \in F$ implies $x \wedge y \in F$.

3. $x \in F$ and $x \leq y$ implies $y \in F$.

4. F is maximal with respect to the properties (1), (2) and (3).

Example In the case of X finite, the ultrafilters in P(X) are in bijective correspondence with the elements of X.

Two constructions

1. Let B be a unital Boolean algebra. Denote by X(B), the *structure space*, the set of all ultrafilters of B. For each $e \in B$ define U_e to be the set of all ultrafilters that contain e. Put

$$\tau = \{U_e : e \in B\}.$$

   Then τ is the basis for a topology on X(S), with respect to which it becomes a Stone space.

2. Let X be a Stone space. Denote by K(X) the set of all clopen subsets of X. Then K(X) is a Boolean algebra.

These two constructions are mutually inverse, up to isomorphism, and lead to a proof of the dual equivalence of categories.

Example It is a theorem of Tarski that up to isomorphism there is exactly one countable atomless Boolean algebra. We call this Boolean algebra the *Tarski algebra*. The Stone space of the Tarski algebra is the *Cantor space*.

2. Etale groupoids

A *groupoid* is a category in which every arrow is an isomorphism.

BUT

- Forget the usual set-theoretic flummery. Our groupoids are just sets.

- Our groupoids are 1-sorted not 2-sorted structures. Everything is an arrow with special arrows called *identities* replacing objects.

- A group is a groupoid with exactly one identity and so is *degenerate*.

Let G be a groupoid with set of identities G_o. Define

$$\mathbf{d}(x) = x^{-1}x \text{ and } \mathbf{r}(x) = xx^{-1}.$$

Define

$$G * G = \{(x, y) \in G \times G \colon \mathbf{d}(x) = \mathbf{r}(y)\},$$

the set of *composable pairs*. Define

$$\mathbf{m} \colon G * G \to G \text{ by } (x, y) \mapsto xy.$$

We say that G is a *topological groupoid* if G is equipped with a topology and $G * G$ with the induced topology from $G \times G$ such that all maps $\mathbf{d}$, $\mathbf{r}$, $x \mapsto x^{-1}$ and $\mathbf{m}$ are continuous.

We will only study a special class of topological groupoids.

A topological groupoid G is said to be *étale* if $\mathbf{d}$ is a local homeomorphism.

WHY ETALE?

The explanation follows from an important observation by Pedro Resende:

if G is étale then $\Omega(G)$ is a monoid.

In fact, $\Omega(G)$ is an example of a *quantale*.

See: P. Resende, *Lectures on étale groupoids, inverse semigroups and quantales*, August, 2006.

Quantales play an important role in our most recent work but not in these lectures (lack of time, not lack of importance).

Take home message: Etale groupoids have a strong algebraic character.

Etale groupoids in nature

1. M. R. Bridson, A. Haefliger, *metric spaces of non-positive curvature*, Springer, 1999.

2. A. L. T. Paterson *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1998.

3. Boolean inverse $\wedge$-monoids and their étale groupoids

We shall generalize the duality between unital Boolean algebra and Stone spaces as follows:

Boolean algebras $\longrightarrow$ class of inverse monoids

Stone spaces $\longrightarrow$ class of étale groupoids

An inverse monoid S is said to be *Boolean* if $E(S)$ is a Boolean algebra, if $a, b \in S$ and $a \sim b$ then $\exists a \vee b$, and if $c \in S$ then $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$. It is said to be a $\wedge$-*monoid* if $\exists a \wedge b$ for all $a, b \in S$.

A *Boolean groupoid* is an étale groupoid whose space of identities is a Stone space. We shall be interested in *Hausdorff Boolean groupoids*.

From groupoids to inverse monoids

This is the easy direction.

Let G be a groupoid. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Let G be a Hausdorff Boolean groupoid. Denote by $\mathsf{K}(G)$ the set of all compact-open local bisections. Then $\mathsf{K}(G)$ is a Boolean inverse $\wedge$-monoid.

In seeing this, it is helpful to bear in mind the following topological results (see Simmons):

- Any closed subset of a compact space is compact.

- Any compact subspace of a Hausdorff space is closed.

From inverse monoids to groupoids

This is the hairy direction.

Let S be a Boolean inverse $\wedge$-monoid. Denote by $\mathsf{G}(S)$ the set of all ultrafilters of S.

To make progress, we need to analyze ultrafilters on inverse monoids. It turns out that they look like *cosets*.

Define an ultrafilter to be *idempotent* if it contains an idempotent.

We need some notation. If $Y \subseteq S$ denote by $Y^{\uparrow}$ the set of all $s \in S$ such that $y \leq s$ for some $y \in Y$.

Lemma An ultrafilter is idempotent if, and only if, it is an inverse submonoid. Each idempotent ultrafilter is of the form $F^{\uparrow}$ where $F \subseteq E(S)$ is a Boolean algebra ultrafilter.

Let $A \subseteq S$ be an ultrafilter. Define

$$\mathbf{d} = (A^{-1}A)^{\uparrow} \text{ and } \mathbf{r} = (AA^{-1})^{\uparrow}.$$

Both of these are idempotent ultrafilters. We now have the following coset form for ultrafilters.

Lemma Let A be an ultrafilter. Then

$$A = (a\mathbf{d}(A))^{\uparrow}$$

where $a \in A$.

Lemma Let A and B be ultrafilters. If $\mathbf{d}(A) = \mathbf{r}(B)$ then $A \cdot B = (AB)^{\uparrow}$ is an ultrafilter. In addition, $\mathbf{d}(A \cdot B) = \mathbf{d}(B)$ and $\mathbf{r}(A \cdot B) = \mathbf{r}(A)$.

Proposition Let S be a Boolean inverse $\wedge$-monoid. Then $(\mathsf{G}, \cdot)$ is a groupoid. The identities are the idempotent ultrafilters.

First non-commutative duality theorem

We now have to endow our groupoid with a topology. For each $s \in S$ define V_s to be the set of all ultrafilters of S that contain s. Put $\tau = \{V_s : s \in S\}$. Then τ is the basis for a Hausdorff topology on $\mathsf{G}(S)$ with respect to which it is a Hausdorff Boolean groupoid.

Theorem For suitable classes of morphisms, the category of Boolean inverse $\wedge$-monoids is dually equivalent to the category of Hausdorff Boolean groupoids.

The functors that effect this duality are

$$S \mapsto \mathsf{G}(S) \text{ and } G \mapsto \mathsf{K}(G).$$

Example The groupoid associated with the finite symmetric inverse monoid $I(X)$ is $X \times X$.

4. Generalizations

I shall just summarize these since they won't be explicitly needed in these lectures.

- Replace Boolean inverse $\wedge$-*monoids* by Boolean inverse $\wedge$-*semigroups*. This means we replace compact by locally compact.

- Replace Boolean inverse $\wedge$-semigroups by Boolean semigroups. This means we lose Hausdorffness.

- Generalize to distributive inverse semigroups. This means we replace ultrafilters by prime filters.

- Replace étale *groupoids* by étale *categories*. This means we replace *inverse* semigroups by *restriction* semigroups.

- Replace étale *topological* categories by étale *localic* categories. This also accommodates *quantales*.

5. How to construct Boolean inverse $\wedge$-monoids

Apart from symmetric inverse monoids, what other examples of Boolean inverse $\wedge$-monoids are there? There are two techniques for constructing them.

1. *AF inverse monoids.* Not discussed here, but the idea is to construct certain colimits of finite direct products of finite symmetric inverse monoids.

2. By completing inverse semigroups. This I shall now describe but since it is technical I shall focus on the main ideas.

Idea

When we constructed C_2 from P_2 we used the *relation*

$$1 = aa^{-1} \vee bb^{-1}.$$

Let's look inside P_2.

What is significant about the way that aa^{-1} and bb^{-1} lie beneath 1?

Answer: everything beneath 1 must meet at least one of aa^{-1} or bb^{-1}.

For the cognoscenti We are, in some sense, rendering abstract the fact that $\{a, b\}$ is a *maximal prefix code*.

Tight coverages

Let S be an inverse semigroup. **Not** necessarily Boolean — this is the whole point — **not** necessarily a monoid (though for simplicity you can assume they are), not necessarily a $\wedge$-monoid (though for simplicity you can assume they are).

Notation Let $s \in S$. We denote by $s^{\downarrow}$ the set of all elements below s.

Let $s \in S$. A *finite* subset $A \subseteq s^{\downarrow}$ is called a *tight cover of* s if for every $t \leq s$ there exists $a \in A$ such that $t \wedge a \neq 0$.

Paraphrase Each element below a must meet at least one element in A in a non-trivial way.

Example In the polycyclic monoid P_2 the set $\{aa^{-1}, bb^{-1}\}$ is a tight cover of 1.

The set of all tight covers of all elements of an inverse semigroup forms what is called the *tight coverage*.

The ideas described here combine independent work by Ruy Exel, Lenz and me.

Because the only covers we shall look at in these lectures are tight covers, I shall just use the word *cover* from now on.

Let S be an inverse semigroup equipped with a coverage. Let T be a distributive inverse monoid. A morphism $\theta\colon S \to T$ is called a *cover-to-join* map if for every $\{a_1, \ldots, a_m\} \subseteq s^{\downarrow}$ a cover we have that

$$\theta(s) = \bigvee_{i=1}^{m} \theta(a_i).$$

Theorem Let S be an inverse semigroup. Then there is a distributive inverse semigroup $\mathsf{D}_t(S)$ and a cover-to-join map $\tau\colon S \to \mathsf{D}_t(S)$ with the property that every cover-to-join map from S factors through τ.

We call $\mathsf{D}_t(S)$ the *tight completion* of S.

Paraphrase The tight completion of S can be regarded as the most general distributive inverse semigroup generated by S and subject to the *relations*

$$s = \bigvee_{i=1}^{m} a_i$$

whenever $\{a_1, \ldots, a_m\}$ is a cover of s.

Remarks

1. If S has a finite number of maximal idempotents $\{e_1, \ldots, e_m\}$ then the tight completion will be a monoid. Thus we don't need to start with a monoid to get a monoid.

2. If S has the property that any two principal order ideals intersect in a finitely generated order ideal then the tight completion will be a $\wedge$-monoid.

3. Under what circumstances will the tight completion be Boolean? We shall say that an inverse semigroup is *pre-Boolean* if its tight completion is Boolean. Necessary and sufficient conditions are known. There are also some useful sufficient conditions. See: M. V. Lawson, Compactable semilattices, *Semigroup Forum* **81** (2010), 187–199.

6. Back to Cuntz inverse monoids

The upshot is that **IF** you start with an inverse *monoid* such that

- Every element above a non-zero idempotent is an idempotent.

- Its semilattice of idempotents forms a "reasonable" tree.

THEN it will be pre-Boolean and its tight completion will be a Boolean inverse $\wedge$-monoid.

. . . such as . . . the polycyclic inverse monoids P_n.

ALL the tight coverage relations can be shown to boil down to **JUST ONE** which, in the case of P_2, is just

$$1 = aa^{-1} \vee bb^{-1}.$$

THUS *the Cuntz inverse monoid C_n is the tight completion of P_n.*

Theorem Under non-commutative Stone duality, the étale groupoid associated with C_n is the same as the one used to construct the Cuntz C^*-algebra (see Renault).

Coverages

There is a general notion of a coverage on an inverse semigroup.

Essentially, inverse semigroup + coverage gives rise to a pseudogroup. Think *generators and relations.*

In the case of tight coverages, we are able to cut down to distributive inverse semigroups.

Link to the next section

Our results put the focus on Boolean inverse $\wedge$-monoids and their groups of units.

Recall that examples of such inverse monoids are the symmetric inverse monoids, and these have interesting groups of units.

III. Groups

1. Motivation: finite symmetric inverse monoids

2. Cantor monoids

1. Motivation: finite symmetric inverse monoids

We begin by returning to symmetric inverse monoids. But here we shall be interested in the *finite* such monoids.

Denote the finite symmetric inverse monoid on n letters by I_n.

We begin with an abstract characterization.

This requires two new ideas.

An inverse semigroup is *fundamental* if the only elements that commute with all idempotents are themselves idempotents.

Remark Fundamental inverse semigroups can be visualized as being inverse semigroups of partial homeomorphisms on a topological space where the idempotents determine the topology.

A Boolean inverse monoid is said to be 0-*simplifying* if there are no non-trivial $\vee$-closed ideals.

We therefore have the following sequence of ever weaker notions

congruence-free, 0-simple, 0-simplifying

Result A Boolean inverse semigroup is congruence-free if, and only if, it is fundamental and 0-simple.

Theorem Let S be a finite Boolean inverse $\wedge$-monoid.

1. There exists a finite discrete groupoid G such that S is isomorphic to $\mathsf{K}(G)$. [Compare with with the usual structure theorem for finite unital Boolean algebras.]

2. If S is fundamental then S is isomorphic to a finite direct product of finite symmetric inverse monoids. We call these *semisimple inverse monoids*.

3. If S is 0-simplifying and fundamental then S is isomorphic to a finite symmetric inverse monoid.

Take home message: 0-simplifying, fundamental Boolean inverse $\wedge$-monoids should be regarded as generalizations of finite symmetric inverse monoids.

Properties of finite symmetric inverse monoids

- Groups of units are finite symmetric groups S_n.

- For $n \geq 5$, the commutator subgroups of S_n are simple.

- The structure of the groups of units is controlled by involutions.

- $I_m \cong I_n$ if, and only if, $S_m \cong S_n$.

- Each element in I_n lies beneath an element of S_n, though not uniquely.

Decomposing involutions

We now consider the group of units S_n from the perspective of I_n.

The atoms of I_n are the partial bijections of the form $x \mapsto y$, where domain and range each contain exactly one element. Each element of I_n is an (orthogonal) join of a finite number of such atoms.

Consider now the involution $(12) \in S_n$ where $n \geq 2$. For concreteness, choose $n = 4$. Then

$$(12) = (1 \mapsto 2) \vee (2 \mapsto 1) \vee (3 \mapsto 3) \vee (4 \mapsto 4).$$

Put $a = 1 \mapsto 2$. Put $e = (3 \mapsto 3) \vee (4 \mapsto 4)$. Then

$$(12) = a \vee a^{-1} \vee e$$

where $a^2 = 0$.

Infinitesimals

A non-zero element a in an inverse semigroup is called an *infinitesimal* if $a^2 = 0$. We have shown above that transpositions in finite symmetric groups may be constructed from infinitesimals.

Lemma Let S be a Boolean inverse $\wedge$-monoid. If a is an infinitesimal then

$$t = a \vee a^{-1} \vee e,$$

where $e = \overline{a^{-1}a}\,\overline{aa^{-1}}$, is an involution above a.

2. Cantor monoids

This section was inspired by, or is a translation of (under non-commutative Stone duality),

Hiroki Matui, Topological full groups of one-sided shifts of finite type, arXiv: 1210.5800v3

Proposition Let S be a countable Boolean inverse $\wedge$-monoid. If S is 0-simplifying then either $E(S)$ is finite or $E(S)$ is the Tarski algebra.

This motivates the following definition.

A *Cantor monoid* is a countable Boolean inverse $\wedge$-monoid whose Boolean algebra of idempotents is a Tarski algebra.

A Boolean inverse monoid S is said to be *piecewise factorizable* if each element s can be written in the form

$$s = \bigvee_{i=1}^{m} g_i s^{-1} s$$

where the g_i are units.

Proposition Every 0-simplifying Cantor monoid is piecewise factorizable.

Thus the elements of such monoids are glued together from restrictions of units.

Let S be a semigroup and e an idempotent. Then eSe is called a *local submonoid*.

Proposition Let S be a 0-simplifying Cantor monoid. Then it is 0-simple if, and only if, every non-zero local submonoid contains a copy of P_2.

Let S be a Boolean inverse $\wedge$-monoid. For each $s \in S$ define

$$\phi(s) = s \wedge 1.$$

This is the largest idempotent below s. Definition due to Leech. Define

$$\sigma(s) = \overline{\phi(s)} s^{-1} s.$$

This is called the *support of* a and σ the *support operator*.

Proposition [Enough involutions] Let S be a fundamental, 0-simplifying Cantor monoid. Then for each ultrafilter $F \subseteq E(S)$ and idempotent $e \in F$ there exists a non-trivial involution t such that $\sigma(t) \in F$ and $\sigma(t) \leq e$.

Two emblematic theorems

The first is Rubinesque. See: M. Rubin, On the reconstruction of topological spaces from their groups of homomorphisms, *TAMS* **312** (1989), 487–538.

Theorem [Spatial realization] Let S and T be two 0-simplifying, fundamental Cantor monoids. Then the following are equivalent

1. S and T are isomorphic.

2. The groups of units of S and T are isomorphic.

Theorem Let S be a 0-simple, fundamental (so congruence-free) Cantor monoid. Then the commutator subgroup of its group of units is simple.

Envoi

- We haven't talked about the Boolean inverse $\wedge$-monoids constructed from finite directed graphs. Their groups are natural generalizations of the groups $G_{n,1}$. See: Graph inverse semigroups: their characterization and completion, with D. Jones, *Journal of Algebra* **409** (2014), 444–473.

- There are Boolean inverse $\wedge$-monoids associated with aperiodic tilings. The nature of their groups is the subject of ongoing work.

- There are clear connections with the work of Hughes on ultrametric spaces.

- What are the inverse monoids associated with Cantor minimal systems?

- Finer classifications probably achieved using homology/cohomology.

Classification problem

Classify the countable Boolean inverse $\wedge$-monoids.

In addition, determine the nature of

- their groups of units
- their associated groupoids
- their associated C^*-algebras

"Only connect! . . . Live in fragments no longer."

E. M. Forster