Higher dimensional generalizations of the Thompson groups via higher rank graphs

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HWU

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What this talk is about

My goal is to show you how to construct groups from certain cancellative monoids. It is possible to replace cancellative monoids by suitable cancellative categories, but *sufficient unto the day is the evil thereof*. The cancellative monoids introduced are of independent interest.

The origins of this talk lie in a couple of papers I wrote back in 2007, after reading a paper by Birget of 2004. I revisited this work in 2019, and more recently first with Alina Vdovina (2020), and then with Aidan Sims and Alina Vdovina (2024). There are also connections with recent work due to Richard Garner.

1. Idea

We want to construct groups.

Groups are abstract versions of groups of bijections.

How should we construct bijections?

Construct bijections by glueing together *partial bijections*.

But, partial bijections can only be glued together if they are *compatible*.

Thus, we can construct bijections by glueing together compatible sets of partial bijections.

The abstract theory of partial bijections is inverse semigroup theory.

Where do partial bijections come from?

One possible source of examples is provided by cancellative monoids. For example, we can multiply by an element on the left — this is a partial bijection, *comme ça*:

if M is a cancellative monoid and $a \in M$ then we get a partial bijection $\lambda_a \colon M \to M$ by defining $\lambda_a(x) = ax$. This partial bijection has domain M and range aM.

The goal of this talk is therefore to show how to construct groups from certain cancellative monoids.

The cancellative monoids in question are certain *one vertex higher rank graphs*.

These really are monoids (not graphs).

2. Inverse semigroups as the abstract theory of partial bijections.

We shall ultimately construct our groups from inverse semigroups which are, in turn, constructed from cancellative monoids.

As groups are to bijections, so inverse semigroups are to partial bijections.

> *Symmetry denotes that sort of concordance of several parts by which they integrate into a whole.* − Hermann Weyl

Inverse semigroups arose by abstracting *pseudogroups of transformations* in the same way that groups arose by abstracting groups of transformations. There were three independent approaches:

- Gordon B. Preston (1925–2015) in the UK;
- Charles Ehresmann (1905–1979) in France;
- Viktor V. Wagner (1908–1981) in the USSR.

They all three converge on the definition of *inverse semigroup*.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Observe that aa^{-1} and $a^{-1}a$ are idempotents.

The idempotents in an inverse semigroup always commute with each other (this is elementary but not easy to prove).

Example: Groups are the inverse semigroups having exactly one idempotent.

If θ is a semigroup homomorphism with domain an inverse semigroup and $\theta(a)$ is an idempotent, then there is an idempotent e such that $\theta(e) = \theta(a)$. (This is a special case of Lallement's lemma).

The image of an inverse semigroup under a semigroup homomorphism is always inverse.

Example: the symmetric inverse monoid

Let X be any non-empty set. Denote by $\mathcal{I}(X)$ the set of all partial bijections of X. This is an example of an inverse semigroup called the *symmetric inverse monoid*.

- The inverse of the partial bijection f is f^{-1}.

- The idempotents are the identity functions on the subsets of X. Thus, if $A \subseteq X$ then the corresponding idempotent is 1_A.

- The product of two idempotents is the idempotent defined on the intersection of their domains of definition (this witnesses the fact that the idempotents commute).

The fact that inverse semigroups really are the abstract theory of partial bijections is expressed by the following which is the analogue of Cayley's theorem.

Theorem [Wagner-Preston] *Every inverse semigroup can be embedded in a symmetric inverse monoid.*

Define $a \leq b$ if $a = ba^{-1}a$. This is a partial order called the *natural partial order*. It has some nice properties:

- If $a \leq b$ and $c \leq d$ then $ac \leq bd$.

- If $a \leq b$ then $a^{-1} \leq b^{-1}$.

- If $a \leq e$ and e is an idempotent then a is an idempotent.

- If $a, b \leq c$ then ab^{-1} and $a^{-1}b$ are idempotents.

Define $a \sim b$, and say that a and b are *compatible*, if $a^{-1}b$ and ab^{-1} are both idempotents.

Example: back to the symmetric inverse monoid

Crucially, we have the following:

- $f \leq g$ if and only if $f \subseteq g$.

- $f \sim g$ if and only if $f \cup g$ is a partial bijection.

Thus we can glue two partial bijections together to get another partial bijection precisely when they are compatible.

3. Groups from inverse semigroups.

Recall that groups are inverse semigroups with exactly one idempotent.

If S is an inverse semigroup, define $a \, \sigma \, b$ if there exists $c \leq a, b$. Observe that σ is a congruence on S.

If e and f are idempotents then $ef \leq e, f$. So, all idempotents are identified by σ.

Theorem *The inverse semigroup S/σ is a group, and if ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$.*

Thus, σ is the most efficient way of getting a group out of an inverse semigroup.

BUT . . .

The problem with the above construction is that if S contains a zero then the group above is trivial.

This suggests that we look at *large* elements of S, which exclude the zero. In this talk, large means the following.

We say that a non-zero idempotent e is *essential* if $ef \neq 0$ whenever f is a non-zero idempotent. We say that the element $a \in S$ is *essential* if both $a^{-1}a$ and aa^{-1} are essential idempotents.

The *essential part*, S^{e}, of the inverse semigroup S consists of all the essential elements of S. It is easy to show that (if non-empty) S^{e} is always an inverse subsemigroup of S.

Define the group associated with the inverse semigroup S as follows:

$$\boxed{\mathscr{G}(S) = S^{e}/\sigma.}$$

From now on, we shall ONLY consider groups constructed from inverse semigroups in the above way.

4. Inverse semigroups from cancellative monoids.

We shall now find a source of examples of inverse semigroups to which we can apply the above constructions. Let M be a cancellative monoid.

An *invertible element* in a monoid is an element x for which there exists an element y such that $yx = 1$ and $xy = 1$.

A monoid in which the only invertible elements are the identities is said to be *conical*.

It is convenient to assume that from now on all our monoids are cancellative and conical.

It is worth noting that we shall build groups from cancellative monoids which are almost entirely devoid of invertible elements.

A subset $R \subseteq M$ is said to be a *right ideal* if $r \in R$ and $a \in M$ implies that $ra \in R$. That is: $RM \subseteq R$.

If $X \subseteq M$ is any subset then XM is the right ideal *generated* by X. If X is a finite set we say that XM is a *finitely generated* right ideal.

We call aM the *principal right ideal* generated by a.

We say that the monoid M is *finitely aligned* if, when $aM \cap bM$ is non-empty, it is always a finitely generated right ideal. This definition is taken from the theory of C^*-algebras but seems to have first been studied witin semigroup theory by Victoria Gould.

Let R_1 and R_2 be right ideals of the monoid M. A function $\theta\colon R_1 \to R_2$ is a *morphism* if $\theta(ra) = \theta(r)a$ for all $a \in M$.

Morphisms are therefore analogous to the homomorphisms between right R-modules in ring theory.

A bijective morphism is called an *isomorphism*.

We say that a non-empty right ideal is *essential* if it intersects every principal right ideal non-trivially.

This will match our earlier definition of ‘essential’.

Theorem *Let M be a cancellative, conical finitely aligned monoid. Then $\mathsf{R}(M)$, the set of all isomorphisms between the finitely generated right ideals of M, is an inverse monoid. The group associated with M is defined to be*

$$\boxed{\mathscr{G}(M) = \mathsf{R}(M)^e/\sigma.}$$

where $\mathsf{R}(M)^e$ is the set of isomorphisms between the essential finitely generated right ideals of M.

We can say more about the structure of the inverse monoid $\mathsf{R}(M)^e$.

An inverse semigroup S is said to be E-*unitary* if $e \leq a$, where e is an idempotent, implies that a is an idempotent.

Lemma *An inverse semigroup S is E-unitary if and only if $\sigma = \sim$.*

Proposition *The inverse monoid $\mathsf{R}(M)^e$ is E-unitary.*

We have shown how to obtain a group from a cancellative monoid.

But to say more about the structure of the group, we have to impose some extra conditions on the cancellative monoid.

5. Projective right ideals.

Let M be a monoid. The elements $a, b \in M$ are said to be *independent* if $aM \cap bM = \varnothing$ otherwise they are said to be *dependent*.

A finite set of independent elements is called a *code*.

A code X is said to be *maximal* if every element of M is dependent on an element of X.

A right ideal generated by a code is said to be *projective*. The study of projective right ideals is due to John Fountain. An essential projective right ideal is precisely that generated by a maximal code.

A monoid M is said to be *strongly finitely aligned* if $aM \cap bM \neq \varnothing$ is projective.

[We can now see the point of *conical* cancellative monoids. If M is a conical cancellative monoid and X and Y are codes then we have that $XM = YM \Leftrightarrow X = Y$. Thus right ideals generated by codes can be labelled by the codes alone. I shall not use this, but it is used in representing the elements of the Thompson groups.]

Theorem *Let M be a conical cancellative strongly finitely aligned monoid. Then* $\mathsf{P}(M)$*, the set of all isomorphisms between the projective right ideals of M, is an inverse monoid. The inverse monoid* $\mathsf{P}(M)^e$ *consists of all the isomorphisms between the projective right ideals generated by maximal codes.*

We now add in one extra assumption which is included in the theorem below.

Theorem *Let M be a cancellative strongly finitely aligned conical monoid. Suppose that every essential finitely generated right ideal contains an essential projective right ideal. Then*

$$\boxed{\mathscr{G}(M) \cong \mathsf{P}(M)^e/\sigma.}$$

With these extra assumptions on our cancellative monoids, our groups are defined in terms of maximal codes.

6. An example.

I shall now apply the general theory above to describe my 2007 paper which only used the theory of free monoids and developed some ideas to be found in Birget.

Let A be any (non-empty) set called in this context an *alphabet*.

Denote by A^* the set of all finite sequences from A. Elements of A^* are called *strings*.

Equipped with concatenation as the product, the set A^* becomes a semigroup. It has an identity, the empty string ε, and so A^* is a monoid. A^* is the *free monoid* on the set A.

This monoid is cancellative and the only invertible element is the identity. Free monoids are therefore cancellative conical monoids.

Let A be an alphabet.

Let u and v be strings. Then $uA^* \cap vA^*$ could be empty. If it is not empty then either u is a *prefix* of v, or v is a prefix of u; this means that $uA^* \cap vA^* = vA^*$, or $uA^* \cap vA^* = uA^*$.

Thus free monoids are finitely aligned.

A code in a free monoid is simply a *finite prefix code.*

Every finitely generated right ideal of a free monoid is generated by a *prefix code.*

Every essential finitely generated right ideal of a free monoid is generated by a *maximal prefix code*.

Let A be an alphabet with at least one element. The group associated with A^* is the group

$$\boxed{\mathscr{G}(A^*) \cong \mathsf{P}(A^*)^e/\sigma.}$$

Theorem *When A has n elements, where n is finite and at least 2, then this group is the Thompson group $G_{n,1}$. Thus when $n = 2$ this group is the Thompson group often denoted by V.*

7. Generalizations of free monoids: one-vertex higher rank graphs.

So matters stood until I came across the paper by Kumjian and Pask (2009).

This showed me how to generalize free monoids and so construct their groups.

We show first how to generalize free monoids. We shall need some notation.

Denote by $\mathbb{N}$ the set of natural numbers under addition and with the usual order.

Denote by $\mathbb{N}^k$ the set of k-tuples of the natural numbers under componentwise addition and order.

We denote the additive identity of $\mathbb{N}^k$ by $\mathbf{0}$.

Let A^* be a free monoid. Then there is a homomorphism

$$\delta\colon A^* \to \mathbb{N}$$

given by $\delta(x) = |x|$, the *length* of the string x.

This homomorphism has a nice property. Suppose that $\delta(x) = m + n$, where $m, n \in \mathbb{N}$. Then there are unique elements u and v of A^* such that $x = uv$ where $\delta(u) = m$ and $\delta(v) = n$.

More generally, ...

A *monoid* M is said to be a *one-vertex higher rank graph* or a k-*monoid* (our preferred term in this context) if there is a homomorphism $\delta\colon M \to \mathbb{N}^k$, called the *degree map*, satisfying the *unique factorization property* (UFP): if $\delta(a) = \mathbf{m} + \mathbf{n}$ then there are unique elements a_1 and a_2 in M such that $a = a_1 a_2$ where $\delta(a_1) = \mathbf{m}$ and $\delta(a_2) = \mathbf{n}$. We call $\delta(x)$ the *degree* of x.

If M is a k-monoid for some k, but we are not particular about the number k, then we shall say that M is a *poly-monoid*.

The only difference with the definition you will find in Kumjian and Pask is that I do not assume that M is countable.

The place of free monoids within the broader theory of poly-monoids is spelled out by the following theorem.

Theorem *The* 1*-monoids are precisely the free monoids. The degree map is just the usual length function.*

N.B. However, if M and N are poly-monoids so too is $M \times N$. Thus the class of poly-monoids is closed under finite direct products. This is not true of free monoids.

How to think about the elements of poly-monoids.

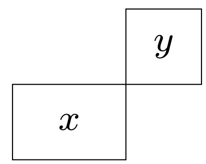
The elements of 1-monoids are strings.

The elements of 2-monoids can be regarded as rectangles. These can be described as Zappa-Szép products of free monoids.

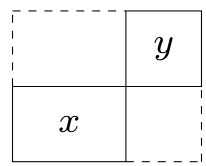
The elements of 3-monoids can be regarded as cuboids etc.

Visualization of the multiplication in 2-monoids

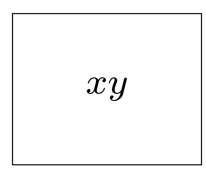
Given x and y, to calculate xy



use UFP to fill in the blanks



and now we get the result xy



It can be proved that, if M is a k-monoid, then:

1. M is cancellative.

2. M is conical.

I will add a couple of further assumptions (familiar to C*-algebra theorists).

A one-vertex higher rank graph M has *no sources* if the map δ is surjective (this rules out the free monoid on no generators).

A one-vertex higher rank graph M is *row finite* if the number of elements of M of degree $\mathbf{m}$ is finite.

Example

Let A be a set.

Then, A^* has no sources means that A consists of at least one element.

In addition, A^* is row finite precisely when A is finite.

We now have the following results:

- A k-monoid which is row finite is strongly finitely aligned.

- A k-monoid which has no sources is such that if $\mathbf{m} \in \mathbb{N}^k$ then the set of all elements of M of degree $\mathbf{m}$ is a maximal code.

- Every finitely generated essential right ideal contains a right ideal generated by a maximal code.

Thus k-monoids which are row finite and have no sources are cancellative monoids to which we may apply our constructions:

Theorem *Let M be a k-monoid which is row finite and hs no sources. Then, we may construct the group $\mathscr{G}(M)$ as* $\mathsf{R}(M)^e/\sigma$ *which is isomorphic with* $\mathsf{P}(M)^e/\sigma$*.*

To understand the group above, we therefore have to understand maximal codes in k-monoids. I do not.

Example

Let A be a two-element alphabet. Then $A^* \times A^*$ is not a free monoid but is a 2-monoid.

The group associated with this monoid is the group $2V$ introduced by Matt Brin.

This is an example of a *higher dimensional Thompson group*.

In fact, there are groups associated with any finite direct product of free monoids.

The following theorem is more advanced and requires a knowledge of étale groupoids.

Theorem *Let M be a k-monoid with no sources and is row finite. If M is also aperiodic and cofinal then $\mathscr{G}(M)$ is a topological full group of an étale groupoid which is Hausdorff, effective and minimal. If $\mathscr{G}(M)$ is, in addition, countably infinite then it is isomorphic to a subgroup of the group of automorphisms of the Cantor set.*

An example (of the above).

We return to our 2007 paper.

Let A be an alphabet with exactly two elements.

The group $\mathscr{G}(A^*)$ is the Thompson group V or $G_{2,1}$.

It is the group of units of a simple Boolean inverse meet-monoid C_2 (called the *Cuntz inverse monoid*).

Under *non-commutative Stone duality*, this is isomorphic to the topological full group of a Hausdorff, effective minimal étale groupoid the identity space of which is the Cantor space.

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