Co-ordinatizing an MV-algebra

by
a Boolean inverse monoid

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This is joint work with Phil Scott and appeared as:

M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *J. Pure Appl. Algebra* **221** (2017), 45–74.

We have also submitted a paper that develops some of the ideas found above:

M. V. Lawson, P. Scott, Characterizations of classes of countable Boolean inverse monoids, arXiv:2204.10033.

There are two generalizations of Boolean algebras

1. *MV-algebras* are to $\aleph_0$-valued propositional logic as Boolean algebras are to classical two-valued propositional logic. They were introduced by C. C. Chang. The 'MV' stands for 'many-valued'.

2. *Boolean inverse monoids* may be used to generalize classical Stone duality by replacing Boolean spaces by a class of étale topological groupoids.

The aim of this talk is to show how these two generalizations of Boolean algebras are related.

The first order of business is therefore to introduce the two classes of algebraic structure that we shall be dealing with.

1. MV-algebras

An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ where $\oplus$ is a binary operation, $\neg$ is a unary operation, and 0 is a constant satisfying the following axioms:

(MVL1) The binary operation $\oplus$ is associative.

(MVL2) The binary operation $\oplus$ is commutative.

(MVL3) 0 is the identity for $\oplus$.

(MVL4) $\neg$ is an involution.

(MVL5) $\neg 0$ is the zero for $\oplus$.

(MVL6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Examples of MV-algebras

1. Every Boolean algebra is an MV-algebra when it is considered with respect to the operations $(B, \vee, \bar{}, 0)$. Axiom (MV6), simply says that

$$(x \to y) \to y = (y \to x) \to x.$$

2. The unit interval $[0, 1]$ becomes an MV-algebra when we define $x \oplus y = \min\{1, x + y\}$, $\neg x = 1 - x$ and 0 is 0.

3. Put $L_n = \{0\frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. This is an MV-subalgebra of the unit interval. It is a finite MV-algebra. When $n = 2$ we get the two-element Boolean algebra.

Some results about MV-algebras

- An element x of an MV-algebra is said to be *idempotent* if

$$x \oplus x = x.$$

  The set of idempotent elements of an MV-algebra forms a Boolean algebra.

- An MV-algebra is a Boolean algebra if and only if every element is an idempotent.

- Every finite MV-algebra is isomorphic to a finite direct product of MV-algebras of the form L_n.

2. Boolean inverse monoids

Our second generalization of Boolean algebras are the Boolean inverse monoids. We begin with an example to motivate the general definition.

Let X be any non-empty set. Denote by $\mathcal{I}(X)$ the set of all bijections between subsets of X. We call the elements of $\mathcal{I}(X)$ *partial bijections*. If X is finite with n elements we write $\mathcal{I}_n$. Here are some facts about $\mathcal{I}(X)$:

- $\mathcal{I}(X)$ is a monoid with zero.
- The equations $f = fgf$ and $g = gfg$ have a unique solution: namely, $g = f^{-1}$.
- The only idempotents have the form 1_A, the identity functions on a subset $A \subseteq X$.
- $1_A 1_B = 1_{A \cap B}$.
- $f^{-1}f$ is the identity on the domain of definition of f.
- $f \subseteq g$ if and only if $f = gf^{-1}f$.
- The set of idempotents forms a Boolean algebra under $\subseteq$.
- If $f, g \in \mathcal{I}(X)$ then $f \cup g \in \mathcal{I}(X)$ if and only if $f^{-1}g$ and fg^{-1} are both idempotents.

A semigroup S is siad to be *inverse* if the equations $x = xyx$ and $y = yxy$ have a unique solution for any $x \in S$. We denote the unique solution by x^{-1}.

The idempotents in an inverse semigroup always commute.

On every inverse semigroup, we may define a relation by $x \leq y$ if and only if $x = yx^{-1}x$ This is a partial order, called the *natural partial order*, and every inverse semigroup is partially ordered with respect to the natural partial order. This is the only partial order we shall consider on an inverse semigroup.

You should regard the elements of an inverse semigroup as being abstract partial bijections.

If x is an element of an inverse semigroup, then we may think of it like this:

$$xx^{-1}\bullet \xleftarrow{\quad x \quad} \bullet x^{-1}x$$

If e and f are idempotents, we say that there is an *arrow from* e *to* f:

$$f\bullet \longleftarrow \bullet e$$

precisely when there is an element x such that $e = x^{-1}x$ and $f = xx^{-1}$.

An inverse monoid is said to be *Boolean* if it satisfies the following conditions:

- If xy^{-1} and $x^{-1}y$ are both idempotents then $x \vee y$ exists.
- If $x \vee y$ exists then $z(x \vee y) = zx \vee zy$ and $(x \vee y)z = xz \vee yz$.
- The set of idempotents forms a Boolean algebra.

Example The finite direct product

$$\mathcal{I}_{n_1} \times \ldots \times \mathcal{I}_{n_s}$$

is a finite Boolean inverse monoid. It is analogous to the finite-dimensional C^*-algebra

$$M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_s}(\mathbb{C});$$

this analogy is used in our proof of the main theorem in finding the Boolean inverse monoid analogue of an *AF C^*-algebra*.

3. Co-ordinatization

Let S be a Boolean inverse monoid. Denote by $\mathsf{L}(S)$ the set of all equivalence classes $[e]$ where e is an idempotent and $f \in [e]$ precisely when there is an arrow from e to f.

Define a partial operation $\oplus$ on $\mathsf{L}(S)$ by

$$[e] \oplus [f] = [e' \vee f']$$

where $e' \in [e]$ and $f' \in [f]$ and $e'f' = 0$ thus e' and f' are orthogonal.

The structure $\mathsf{L}(S)$ is an *effect algebra*, the definition of which need not detain us here.

We would like to complete $\oplus$ to an everywhere defined operation.

We need to make two further assumptions about our Boolean inverse monoid S in order to make the above operation everywhere defined:

1. Every element of S is below a unit of S.

2. The principal ideals of S form a lattice.

A Boolean inverse monoid is called a *Foulis monoid* if it satisfies conditions (1) and (2) above.

Let S be a Foulis monoid.

Define

$$\neg[e] = [\bar{e}].$$

This makes sense by (1) above.

If e and f are idempotents, then we write $[e] \wedge [f]$ for the equivalence class $[i]$ of the idempotent i so that $SeS \cap SfS = SiS$.

Define

$$[e] \oplus [f] = [e] \oplus ([\bar{e}] \wedge [f]).$$

Despite appearances, this can be shown to make sense by (1) and (2) above.

Proposition *For every Foulis monoid S,* L(S)*, with the above definitions, is an MV-algebra.*

We say that an MV-algebra is *co-ordinatizable* if it is isomorphic to L(S) for some Foulis monoid S.

We can now state our main theorem.

Theorem *Every countable MV-algebra can be co-ordinatized.*

Example $\mathcal{I}_n$ is a Foulis monoid. The finite direct products

$$\mathcal{I}_{n_1} \times \ldots \times \mathcal{I}_{n_s}$$

co-ordinatize the finite MV-algebras.

4. Extensions and further questions

- Fred Wehrung (Theorem 5.2.10 page 164) proved that every MV-algebra is co-ordinatized by some Foulis monoid.

- L(S) is a Boolean algebra if and only if S is such that every idempotent is central.

- Which effect algebras are co-ordinatized by Boolean inverse monoids?

References

1. O. Bratteli, Inductive limits of finite dimensional C^*-algebras, *Trans. Amer. Math. Soc.* **60** (1972), 195–234.

2. C. C. Chang, Algebraic analysis of many valued logic, *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

3. R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Kluwer, Dordrecht, 2000.

4. B. Jacobs, New directions in categorical logic, for classical, probabilistic and quantum logic, *Log. Methods Comput. Sci.* **11** (2015), 1–76.

5. M. V. Lawson, Non-commutative Stone duality, arXiv:2207.02686.

6. M. V. Lawson, *Inverse semigroups*, World Scientific, 1998.

7. M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *J. Pure Appl. Algebra* **221** (2017), 45–74.

8. M. V. Lawson, P. Scott, Characterizations of classes of countable Boolean inverse monoids, arXiv:2204.10033.

9. D. Mundici, Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986), 15–63.

10. D. Mundici, Logic of infinite quantum systems, *Int. J. Theor. Phys.* **32** (1993), 1941–1955.

11. D. Mundici, G. Panti, Extending addition in Eliot's local semigroup, *J. Funct. Anal.* **117** (1993), 461–472.

12. F. Wehrung, *Refinement monoids, equidecomposability types, and Boolean inverse semigroups*, LNM 2188, 2017.