# FROM GROUPOIDS TO GROUPS

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## Background

Interesting papers are appearing in which groups are constructed as topological full groups of étale groupoids.

### Examples

- H. Matui, Homology and topological full groups of étale groupoids on totally disconnected spaces, *Proc. London Math. Soc.* **104** (2012), 27–56.

- H. Matui, Topological full groups of one-sided shifts of finite type, *J. reine angew. Math.* **705** (2015), 35–84.

- V. Nekrashevych, Simple groups of dynamical origin, *Ergod. Th. & Dynam. Sys.*, published online 2017.

These papers are interested in constructing infinite simple groups (amongst other things).

The origins of the ideas they develop lie in the theory of dynamical systems, such as work by Giordano, Putnam and Skau and going back to the late 50s early 60s and papers by Dye.

**The goal of this talk**

To describe the setting from which topological full groups emerge, rather than to focus on interesting results about such groups, per se.

## 1. Etale groupoids

We shall regard groupoids as algebraic structures with a subset of *identities*. If $G$ is a groupoid, its set of identities if $G_o$.

Philip Higgins adopted just such an approach in his applications of groupoid theory to group theory.

### Examples

1. Groups are the groupoids with exactly one identity.

2. Equivalence relations can be regarded as principal groupoids; the *pair groupoid* $X \times X$ is a special case.

3. From a group action $G \times X \to X$ we get the *transformation groupoid* $G \ltimes X$.

A *topological groupoid* is a groupoid $G$ equipped with a topological structure in which both multiplication and inversion are continuous.

A topological groupoid is said to be *étale* if the domain map is a local homeomorphism.

WHY ETALE?

If $X$ is a topological space, denote by $\Omega(X)$ the lattice of all open sets of $X$.

**Theorem** [Resende, 2006] *Let $G$ be a topological groupoid. Then $G$ is étale if and only if $\Omega(G)$ is a monoid.*

- Etale groupoids are topological groupoids with an algebraic alter ego.
- Etale groupoids should be viewed as generalized spaces (Kumjian, Crainic and Moerdijk . . . .)

## 2. Classical Stone duality

A *Boolean space* is a 0-dimensional, compact Hausdorff space.

**Theorem** [Stone, 1937]

1. *Let $S$ be a Boolean space. Then the set* B($S$) *of clopen subsets of $S$ is a Boolean algebra.*

2. *Let $A$ be a Boolean algebra. Then the set* X($A$) *of all ultrafilters of $A$ can be topologized in such a way that it becomes a Boolean space. It is called the* Stone space *of $A$.*

3. *If $S$ is a Boolean space then $S \cong$* XB($S$).

4. *If $A$ is a Boolean algebra then $A \cong$* BX($A$).

## Examples

1. Up to isomorphism, there is exactly one countable, atomless Boolean algebra. It is innominate so I call it the *Tarski algebra*. The Stone space of the Tarski algebra is the Cantor space.

2. The Stone space of the powerset Boolean algebra P($X$) is the Stone-Čech compactification of the discrete space $X$.

## 3. Non-commutative Stone duality

A *Boolean groupoid* is an étale groupoid whose space of identities is a Boolean space.

A *Boolean inverse monoid* is an inverse monoid satisfying the following conditions:

1. The set of idempotents forms a Boolean algebra under the natural partial order.

2. Compatible pairs of elements have a join.

3. Multiplication distributes over the compatible joins in (2).

If only (2) and (3) hold, have a *distributive inverse monoid*.

Recall that . . .

A semigroup $S$ is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

An inverse semigroup $S$ is equipped with two important relations.

$s \leq t$ is defined if and only if $s = te$ for some idempotent $e$. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

$s \sim t$ if and only if $st^{-1}$ and $s^{-1}t$ both idempotents. Called the *compatibility relation*. It controls when pairs of elements are *eligible* to have a join.

**Example** Symmetric inverse monoids $I(X)$ are also the prototypes of Boolean inverse monoids.

Let $G$ be a groupoid. A *partial bisection* is a subset $A \subseteq G$ such that $A^{-1}A, AA^{-1} \subseteq G_o$.

Let $G$ be a Boolean groupoid. The set of compact-open partial bisections of $G$ is denoted by $\mathsf{B}(G)$.

Let $S$ be a Boolean inverse monoid. The set of ultrafilters of $S$ is denoted by $\mathsf{G}(S)$.

**Theorem** [Lawson & Lenz, Resende]

1. *Let $G$ be a Boolean groupoid. Then* $\mathsf{B}(G)$ *is a Boolean inverse monoid.*

2. *Let $S$ be a Boolean inverse monoid. Then* $\mathsf{G}(S)$ *is a Boolean groupoid, called the* Stone groupoid *of $S$.*

3. *If $G$ is a Boolean groupoid then $G \cong$* $\mathsf{GB}(G)$.

4. *If $S$ is a Boolean inverse monoid then $S \cong$* $\mathsf{BG}(S)$.

The topological full group of the Boolean groupoid $G$ is just the group of units of its associated Boolean inverse monoid.

Boolean inverse monoids are ‘ring-like’ with the partial join operation being analogous to the addition in a ring.

Wehrung (2017) proved they form a variety and have a Mal’cev term.

Thus topological full groups arise as groups of units of ring-like algebraic structures.

## Dictionary

| **Algebra** | **Geometry** |
|---|---|
| Countable | Second countable |
| Fundamental | Effective |
| 0-simplifying | Minimal |
| Binary meets | Hausdorff |

*Fundamental* means the only elements centralizing the idempotents are idempotents.

0-*simplifying* means it has no, non-trivial order ideals closed under compatible joins.

*Effective* if the interior of the isotropy groupoid is the space of identities.

*Minimal* means there are no, non-trivial open invariant subsets.

**Proposition** *A Boolean inverse monoid is simple if and only if it is fundamental and* 0*-simplifying.*

**Theorem** [The Tarski dichotomy] *Let* $S$ *be a countable, simple Boolean inverse monoid. Then it is either finite, and isomorphic to a finite symmetric inverse monoid, or it is infinite and atomless and so its Boolean algebra of idempotents is a Tarski algebra.*

**Proposition** *The Stone groupoid of the finite symmetric inverse monoid* $I(X)$ *is the pair groupoid* $X \times X$*.*

Thus topological full groups are *morally* (infinite) generalizations of finite symmetric groups.

A *Tarski* monoid is a countably infinite, atomless Boolean inverse monoid.

A Boolean inverse monoid is said to be a *meet monoid* if it has all binary meets.

The following is a version of Matui's spatial realization theorem.

**Theorem** *Let $S$ and $T$ be simple Tarski meet monoids. Then the following are equivalent:*

1. *$S$ and $T$ are isomorphic.*

2. *The Stone groupoids of $S$ and $T$ are isomorphic.*

3. *The groups of units of $S$ and $T$ are isomorphic.*

## Remarks

Neither Matui nor Nekrashevych mention inverse semigroups, but both carry out calculations in the Boolean inverse monoid associated with a Boolean groupoid, since the following are synonyms for compact-open local bisection:

- Matui refers to *compact open $G$-set*.
- Nekrashevych refers to *bisection*.

Here is an argument from Matui rendered into Boolean inverse monoid language. A non-zero element is called an *infinitesimal* if $a^2 = 0$. Put $e = aa^{-1} \vee a^{-1}a$. Then $g = a \vee a^{-1} \vee \bar{e}$ is an involution, called a *special involution*.

Define $\mathsf{Sym}(S)$ to be ths subgroup of the group of units of $S$ generated by the special involutions.

**Theorem** *Let $S$ be a simple Tarski meet monoid. Then each element $s \in S$ can be written*

$$s = \bigvee_{i=1}^{m} g_i e_i$$

*where the $e_i$ are idempotents and $g_i \in \mathsf{Sym}(S)$.*

## 4. From groupoids to groups

The origins of this theory lie in a specific computation first carried out in 2004.

Mark V. Lawson, The polycyclic monoids $P_n$ and the Thompson groups $V_{n,1}$, *Communications in algebra* **35** (2007), 4068–4087.

The polycyclic inverse monoid $P_n$, where $n \geq 2$, is defined by the following inverse monoid presentation

$$P_n = \langle a_1, \ldots, a_n | a_i^{-1} a_i = 1, a_i^{-1} a_j = 0 \text{ for } i \neq j \rangle.$$

The *Cuntz monoid* $C_n$, where $n \geq 2$, is a simple Tarski meet monoid whose group of units is the Thompson group $V_n$.

In modern terminology, $C_n$ is constructed from $P_n$ by completing $P_n$ to a distributive inverse monoid and then factoring out by the relation

$$1 = \bigvee_{i=1}^{n} a_i a_i^{-1}.$$

But in the above paper, this was accomplished by manipulating prefix codes and maximal codes.

Bleak & Quick: $V_{2,1} = \mathsf{Sym}(C_n)$.

# References


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