Non-commutative Stone duality

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I use categories but am not a category theorist (Disclaimer).

I will have questions for you, too.

On a point of notation, I compose functions from right-to-left.

Categories arise in two ways in this talk:

1. Big categories of structures and arrows.

2. Small categories which should be regarded as generalizations of monoids. They are always 1-sorted.

1. Classical/Commutative Stone duality

A topological space X is called a *Boolean space* if it is compact, Hausdorff and 0-dimensional (that is, it has a base of clopen sets).

Theorem: Classical/Commutative Stone duality. (Stone).

1. *With each Boolean algebra B, we can associate a Boolean space* $\mathsf{X}(B)$, *called the* Stone space *of B. It is a set of ultrafilters.*

2. *With each Boolean space X, we can associate a Boolean algebra*, $\mathsf{B}(X)$, *of clopen subsets.*

3. $B \cong \mathsf{B}(\mathsf{X}(B))$ *for each Boolean algebra B.*

4. $X \cong \mathsf{X}(\mathsf{B}(X))$ *for each Boolean space X.*

5. *The category of Boolean algebras is dually equivalent to the category of Boolean spaces.*

2. Idea behind non-commutative Stone duality

We shall generalize this duality as follows:

1. Replace the Boolean algebra by an inverse monoid which has a ‘Boolean character’.

2. Replace the topological space by a topological groupoid which generalizes a Boolean space. This idea originated in the theory of C^*-algebras.

3. Boolean inverse monoids

An *inverse monoid* is a monoid in which for each element s there is a unique element t such that $s = sts$ and $t = tst$. We usually denote t by s^{-1} and refer to the *inverse* of s.

The idempotents in an inverse monoid commute with each other.

The best example of an inverse monoid is the *symmetric inverse monoid* $\mathcal{I}(X)$ consisting of all partial bijections of the set X.

The idempotents are the identity functions defined on the subsets of X.

Partial bijections can be ordered.

If f and g are partial bijections of the set X then we write $f \leq g$ if the restriction of g to the domain of definition of f is f itself.

This can be defined purely algebraically:

$$f \leq g \text{ iff } f = gf^{-1}f.$$

This motivates the definition of the *natural partial order* on an arbitrary inverse monoid:

$$s \leq t \text{ iff } s = ts^{-1}s.$$

Inverse monoids are partially ordered with respect to the natural partial order.

If $s \leq t$ then $s^{-1} \leq t^{-1}$.

An important property: if $a \leq c$ and $b \leq c$ then ab^{-1} and $a^{-1}b$ are idempotents. Thus, a necessary condition for elements a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotents.

This leads to the *compatibility relation* $a \sim b$ defined by the condition that $a^{-1}b$ and ab^{-1} be idempotents.

An inverse monoid is said to be *Boolean* if it satisfies three conditions:

1. The idempotents form a Boolean algebra w.r.t. the natural partial order.

2. If $s \sim t$ then $s \vee t$ exists.

3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for any $u \in S$.

A *morphism* $\theta\colon S \to T$ of Boolean inverse monoids is a monoid homomorphism that preserves binary joins.

A morphism of Boolean inverse monoids is said to be *proper* if each $t \in T$ we can write $t = \bigvee_{i=1}^{n} t_i$ where $t_i \leq \theta(s_i)$ for some $s_i \in S$.

A morphism of Boolean inverse monoids is said to be *weakly meet preserving* if $t \leq \theta(a), \theta(b)$ there exists $c \leq a, b$ such that $t \leq \theta(c)$.

A morphism of Boolean inverse monoids is said to be *callitic* if it is both proper and weakly meet preserving. The rationale for this definition will be made later.

4. Boolean groupoids

A topological groupoid is said to be *étale* if domain and range maps are local homeomorphisms and the space of identities is open.

Pedro Resende proved that if G is an étale groupoid then $\Omega(G)$ (open sets) is a monoid.

A *Boolean groupoid* is an étale groupoid whose space of identities is a Boolean space.

5. The duality

Let G be a groupoid. A *local bisection* is a subset $A \subseteq G$ such that if $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ (resp. $\mathbf{r}(a) = \mathbf{r}(b)$) implies that $a = b$.

Proposition *Let G be a Boolean groupoid. Then the set of all compact-open local bisections* $\mathsf{KB}(G)$ *is a Boolean inverse monoid.*

Let S be a Boolean inverse monoid.

If X is any subset of S then $X^{\uparrow}$ is the set of all elements above an element of X.

A subset $A \subseteq S$ is called a *filter* if for all $a, b \in A$ there exists $c \in A$ such that $c \leq a, b$, and if $a \in A$ and $a \leq b$ then $b \in A$. It is *proper* if $0 \notin A$. A maximal proper filter is called an *ultrafilter*.

Denote by $\mathsf{G}(S)$ the set of all ultrafilters of S.

Proposition *Let S be a Boolean inverse monoid. Then $\mathsf{G}(S)$, the Stone groupoid of S, is a Boolean groupoid.*

Here is some insight into this result.

If A is an ultrafilter define

$$\mathbf{d}(A) = (A^{-1}A)^{\uparrow} \text{ and } \mathbf{r}(A) = (AA^{-1})^{\uparrow}.$$

Let A and B be ultrafilters. Define

$$A \cdot B = (AB)^{\uparrow} \text{ if } \mathbf{d}(A) = \mathbf{r}(B).$$

$(\mathsf{G}(S), \cdot)$ is a groupoid.

Let A be an ultrafilter that contains an idempotent. We shall call these *idempotent ultrafilters*. Then $A \cap E(S)$ is an ultrafilter in the Boolean algebra $E(S)$.

Let A be an arbitrary ultrafilter. Then $A = (a\mathbf{d}(A))^{\uparrow}$ where $\mathbf{d}(A)$ is an idempotent ultrafilter.

We can say that (approximately) every ultrafilter in S is a coset of an ultrafilter in $E(S)$.

We shall consider functors $\alpha\colon G \to H$ between Boolean groupoids which are continuous covering functors that are also *coherent* in the sense that the inverse image of compact-open sets are compact-open.

Theorem: Non-commutative Stone duality I. (Lawson and Lenz).

1. *$S \cong \mathsf{KB}(\mathsf{G}(S))$ for each Boolean inverse monoid S.*

2. *$G \cong \mathsf{G}(\mathsf{KB}(G))$ for each Boolean groupoid G.*

3. *The category of Boolean inverse monoids and callitic morphisms is dually equivalent to the category of Boolean groupoids and coherent continuous covering functors.*

First question

Let $\theta \colon S \to T$ be a callitic morphism between Boolean inverse monoids. This induces a coherent continuous covering functor $\mathsf{G}(T) \to \mathsf{G}(S)$. How should we denote this functor?

Second question

Let $\alpha \colon G \to H$ be a coherent continuous covering functor between Boolean groupoids. This induces a callictic morphism $\mathsf{KB}(H) \to \mathsf{KB}(G)$. How should we denote this morphism?

The following table is part of a dictionary that translates beween Boolean inverse monoids and Boolean groupoids:

Boolean inverse monoid	**Boolean groupoid**
Group of units of monoid	Topological full group
Countable	Second-countable
Tarski algebra of idempotents	Cantor space of identities
Semisimple	Discrete
Meet monoid	Hausdorff
Fundamental	Effective
Basic	Principal and Hausdorff
0-simplifying	Minimal
Simple	Minimal and effective

We, Franceso Tesolin (PhD student), Ganna Kudryavtseva and me have generalized the duality.

What corresponds to morphisms between Boolean inverse monoids?

Define a *covering relational functor* as follows: these are sub-groupoids $\phi \subseteq G \times H$ between topological groupoids satisfying the following properties:

1. For each identity $e \in G$ there exists a unique identity $f \in H$ such that $(e, f) \in \phi$.

2. If $(a, c), (b, c) \in \phi$ and $\mathbf{d}(a) = \mathbf{d}(b)$ then $a = b$.

3. If $(e, f) \in \phi$, where e and f are identities, and $\mathbf{d}(t) = f$ then there exists s such that $\mathbf{d}(s) = e$ and $(s, t) \in \phi$.

4. If V is open in H then $\phi^{-1}(V)$ is open in G.

Third question

What should we call covering relational functors?

Theorem *The category of Boolean inverse monoids and their morphisms is dually equivalent to the category of coherent covering relational functors and Boolean groupoids.*

The obvious question is whether we can say more.

Can this result be generalized?

Can étale correspondences be included in a dual equivalence?

Extending the duality beyond inverse monoids to other classes of monoids.

6. Non-commutative Stone duality: beyond inverse monoids

We now replace groupoids by categories.

We say that a category is *étale* if its domain and range maps are both local homeomorphisms and the space of identities is open. A category is said to be *Boolean* if it is étale and the space of identities is a Boolean space.

We replace inverse monoids by bi-restriction monoids (see next slide). One approach to understanding these semigroups is that they are defined by axiomatizing the behaviour of the idempotents $s^{-1}s$ and ss^{-1} in an inverse semigroup.

We define a monoid S to be a *right restriction monoid* if it is equipped with a unary operation $a \mapsto a^*$ satisfying the following axioms:

(RR1) $(s^*)^* = s^*$.

(RR2) $(s^*t^*)^* = s^*t^*$.

(RR3) $s^*t^* = t^*s^*$.

(RR4) $ss^* = s$.

(RR5) $(st)^* = (s^*t)^*$.

(RR6) $t^*s = s(ts)^*$.

Those elements a such that $a^* = a$ are called *projections*. *The element a^* in fact axiomatizes the domain of definition of a partial function.*

We define a *left restriction monoid*, dually, and use $a \mapsto a^+$ for the unary operation.

A *bi-restriction monoid* is a monoid which is both a left and right restriction monoid and the sets of projections are the same.

Let S be a bi-restriction monoid. Define

$$y \leq x \text{ iff } y = xy^{*} \text{ equivalently } y = y^{+}x.$$

This is a partial order with respect to which the monoid is partially ordered. This is called the *natural partial order*.

A bi-restriction monoid is said to be *Boolean* if it satisfies three conditions:

1. The projections form a Boolean algebra wrt the natural partial order.

2. If $st^{*} = ts^{*}$ and $s^{+}t = t^{+}s$ then $s \vee t$ exists.

3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for any $u \in S$.

Theorem: Non-commutative Stone duality II. (Kudryavtseva and Lawson).

1. *With each Boolean bi-restriction monoid S, we can associate a Boolean category* $\mathsf{C}(S)$, *called the* Stone category *of S.*

2. *With each Boolean category C, we can associate a Boolean bi-restriction monoid,* $\mathsf{KB}(C)$, *of compact-open local bisections.*

3. $S \cong \mathsf{KB}(\mathsf{C}(S))$ *for each Boolean bi-restriction monoid S.*

4. $C \cong \mathsf{C}(\mathsf{KB}(G))$ *for each Boolean category C.*

7. Non-commutative Stone duality: beyond the beyond

We now replace étale categories by *domain-étale* categories where we only require the domain map to be a local homeomorphism.

A *Boolean domain-étale category* is a domain-étale category whose space of identities is a Boolean space.

Let S be a right restriction monoid. Define

$$y \leq x \text{ iff } y = xy^*.$$

This is a partial order with respect to which the monoid is partially ordered. This is called the *natural partial order*.

A right restriction is said to be *Boolean* if it satisfies three conditions:

1. The projections form a Boolean algebra wrt the natural partial order.

2. If $st^* = ts^*$ then $s \vee t$ exists.

3. If $s \vee t$ exists then $u(s \vee t) = us \vee ut$ and $(s \vee t)u = su \vee tu$ for all $u \in S$.

Theorem: Non-commutative Stone duality III. (Cockett and Garner).

1. *With each Boolean right restriction monoid S, we can associate a Boolean domian-etale category* $\mathsf{C}(S)$*, called the* Stone category *of S.*

2. *With each Boolean domain-etale category C, we can associate a Boolean right restriction monoid,* $\mathsf{KS}(C)$*, of compact-open local* sections.

3. $S \cong \mathsf{KS}(\mathsf{C}(S))$ *for each Boolean right restriction monoid S.*

4. $C \cong \mathsf{C}(\mathsf{KS}(C))$ *for each Boolean domain-etale category C.*

8. In conclusion . . .

- Garner showed that the Boolean right restriction monoids are intimately connected with those varieties (in the sense of universal algebra) which are **Cartesian closed**.

- The work of Cockett and Garner suggests that we may generalize non-commutative Stone duality further, perhaps by using some ideas of Resende.

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