Pseudogroups, Boolean inverse monoids and étale groupoids

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Introductions

1. *Semigroups* seem to be useful tools in studying dynamical systems. The semigroups we consider here are different from the Ellis semigroups.

2. Our work deals with inverse semigroups. These arise in studying certain C^*-algebras. First, in the work of Renault (1980) and then in that of Kellendonk (1997), as the *tiling semigroup* of a tiling, and then in the work of Lenz (2002). Steinberg, Margolis and Lawson rendered Lenz's work purely algebraic. Patson's book (1998) also played an important role.

3. Why are inverse semigroups related to topological groupoids and so to C^*-algebras? Classical Stone duality, when generalized to a non-commutative setting, provides the reason.

Motivation

> "Symmetry denotes that sort of concordance of several parts by which they integrate into a whole." – Hermann Weyl

As groups are algebraic tools for studying symmetry, so inverse semigroups are tools for studying partial symmetry. Symmetry is more than groups.

Our main goal is to investigate the role of inverse semigroups in studying non-classical symmetries such as self-similarity and the symmetry phenomena that arise in aperiodic tilings. The role of inverse semigroups in C^*-algebras à la Renault has been crucial to this.

1. Pseudogroups of transformations

Let X be a topological space. A set S of partial homeomorphisms between all the open subsets of X is called a *pseudogroup of transformations* if it satisfies the following axioms:

(PT1) Closed under products.

(PT2) Closed under inverses.

(PT3) (Completeness) Closed under all non-empty *compatible* unions.

- Associated with the work of Lie, Elie Cartan and Veblen & Whitehead and the foundations of differential geometry.

- Often replaced by an associated étale groupoid of germs.

Example: the symmetric inverse monoid

Let X be a set equippped with the discrete topology. Denote by $I(X)$ the set of all partial bijections of X. This is an example of a pseudogroup called the *symmetric inverse monoid*. If X is finite with n elements denote $I(X)$ by I_n.

Formalizations

Groups of transformations led to the abstract definiton of groups. How can one abstract the notion of pseudogroups of transformations?

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Vagner (1908–1981) in the USSR.

They all three converge on the definition of ‘inverse semigroup’.

Inverse semigroups

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Theorem [Vagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

We cannot quite say that inverse semigroups are the abstractions of pseudogroups because we have not dealt with completeness.

The natural partial order

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup.*

It is called the *natural partial order*.

Example In symmetric inverse monoids the natural partial order is nothing other than the restriction ordering on partial bijections.

Technicalities . . .

Let S be an inverse semigroup. Elements of the form $a^{-1}a$ and aa^{-1} are idempotents. Denote by $E(S)$ the set of idempotents of S.

Remarks

1. $E(S)$ is a commutative subsemigroup.

2. $E(S)$ is an order ideal of S.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A non-empty subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

Example: symmetric inverse monoids

The idempotents in $I(X)$ are the identity functions defined on the subsets of X.

Denote them by 1_A where $A \subseteq X$, called *partial identities*. Then

$$1_A \leq 1_B \Longleftrightarrow A \subseteq B$$

and

$$1_A 1_B = 1_{A \cap B}.$$

An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

We now have an abstract notion of a pseudogroup of transformations.

The idempotents of a pseudogroup form a *frame*. That is, a complete infinitely distributive lattice.

In the case of transformation pseudogroups the idempotents are just the partial identities on the open subsets. Thus the topology of the space is encoded in the set of idempotents.

Pseudogroups and étale groupoids

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms. Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

This establishes a link with *quantales* that will not be discussed here.

Theorem 1 [Lawson & Lenz and Resende] *There is an adjunction between the category of pseudogroups and the dual of the category of étale topological groupoids.*

The proof relies on two concepts: completely prime filters and open local bisections.

- Pseudogroups $\rightarrow$ completely prime filters $\rightarrow$ étale groupoids

- Etale groupoids $\rightarrow$ open local bisections $\rightarrow$ pseuodgroups

A space is *sober* if its open sets determine points. A pseudogroup is *spatial* if its elements are determined by completely prime filters.

Corollary 2 [Lawson & Lenz] *The category of spatial pseudogroups is equivalent to the dual of the category of sober étale topoological groupoids.*

The above theorem is the main tool in what follows.

2. Boolean inverse semigroups

Pseudogroups appear in the rest of the talk in a disguised form which we now explain.

Let S be a pseudogroup. An element $a \in S$ is said to be *finite* if $a = \bigvee_{i \in I} a_i$ implies $a = \bigvee_{i=1}^{n} a_i$ for some n (and possible relabelling).

Denote the set of finite elements of S by $\mathsf{K}(S)$. If this is a distributive inverse semigroup we say that S is *coherent*.

Theorem 3 [Lawson & Lenz] *The category of distributive inverse semigroups is equivalent to the category of coherent pseudogroups.*

A distributive inverse semigroup is said to be *Boolean* if its set of idempotents forms a Boolean algebra.

Why are Boolean inverse semigroups of especial interest?

Theorem [Paterson, Wehrung] *Let S be a subsemigroup of a ring with involution R such that S is an inverse semigroup with respect to the involution. Then there is a Boolean inverse semigroup T such that $S \subseteq T \subseteq R$.*

The above result is significant when viewing inverse semigroups in relation to C^*-algebras.

Theorem 4 [Lawson & Lenz] *Every inverse semigroup can be embedded in a universal Boolean inverse semigroup.*

3. Theorems on Boolean inverse monoids

commutative	non-commutative
frames	pseudogroups
distributive lattices	distributive inverse monoids
Boolean algebras	Boolean inverse monoids

The monoids in the right-hand column are both non-idempotent and non-commutative.

We view the theory of Boolean inverse monoids as that of non-commutative Boolean algebras.

A *Boolean inverse* $\wedge$*-monoid* is a Boolean inverse monoid in which each pair of elements has a meet.

Example Symmetric inverse monoids are Boolean inverse $\wedge$-monoids.

Key definitions

- An inverse monoid is *factorizable* or *unit regular* if each element is below an element in the group of units. Example: symmetric inverse monoids are factorizable if and only if they are finite.

- An inverse semigroup is *fundamental* if the only elements that centralize the idempotents are themselves idempotents. Example: symmetric inverse monoids are fundamental.

- A $\vee$-*ideal* in a Boolean inverse monoid is an ideal closed under finite compatible joins. A Boolean inverse monoid is 0-*simplifying* if it contains no non-trivial $\vee$-ideals. Example: symmetric inverse monoids are 0-simplifying.

Why fundamental?

Theorem [Vagner] *An inverse semigroup is fundamental if and only if it is isomorphic to an inverse semigroup of partial homeomorphisms between the open subsets of a* T_0 *space where the domains of definition of the elements form a basis for the space.*

- Fundamental inverse semigroups should therefore be viewed as inverse semigroups of partial homeomorphisms.

- Each inverse semigroup is an extension of an inverse semigroup with central idempotents by a fundamental one; inverse semigroups with central idempotents are presheaves of groups.

- Being fundamental or being 0-simplifying are both different kinds of ‘simplicity’.

Finite Boolean algebras can be completely and easily classified. So, too, can finite Boolean inverse monoids in an analogous way.

Theorem 5 [Lawson]

1. The finite 0-simplifying, fundamental Boolean inverse monoids are precisely the finite symmetric inverse monoids.

2. The finite fundamental Boolean inverse monoids are precisely the finite direct products of finite symmetric inverse monoids.

3. The finite Boolean inverse monoids are isomorphic to the inverse monoids of local bisections of finite discrete groupoids.

We call finite fundamental Boolean inverse monoids *semisimple*. They have the form $I_{n_1} \times \ldots \times I_{n_r}$.

They are therefore the Boolean inverse monoid analogues of finite dimensional C^*-algebras.

Theorem 6 [Lawson & Scott]

1. The countable, locally finite factorizable fundamental Boolean inverse monoids are precisely the direct limits of semisimple inverse monoids.

2. They can be classified by means of dimension groups and constructed using Bratteli diagrams.

3. They are factorizable (and fundamental) and their groups of units are direct limits of finite direct products of finite symmetric groups and block diagonal maps.

It is natural to call such inverse monoids *AF*.

Let S be a Boolean inverse monoid. Define an equivalence relation on $E(S)$ by $e \equiv f$ if and only if $e = a^{-1}a$ and $f = aa^{-1}$ for some $a \in S$. Equivalence class containing e denoted by $[e]$. Denote by $\mathsf{T}(S)$ the set $\{[e] : e \in E(S)\}$. Define partial addition on $\mathsf{T}(S)$ by $[e] \oplus [f] = [e' \vee f']$ if there exist $e' \equiv e$ and $f' \equiv f$ such that $e'f' = 0$.

The structure $\mathsf{T}(S)$ is the inverse monoid version of a construction used by Elliott in 1976 ('abelian local semigroups').

Proposition 7 [Lawson & Scott]

1. $\mathsf{T}(S)$ *is a* partial commutative monoid.

2. $\mathsf{T}(S)$ *is an* effect algebra *if and only if S is factorizable.*

3. $\mathsf{T}(S)$ *is an* MV algebra *if and only if S is factorizable and $S/\mathscr{J}$, the poset of principal ideals, is a lattice.*

In lieu of a definition: MV algebras are to multiple-valued logic as Boolean algebras are to classical two-valued logic.

Theorem 8 [Lawson & Scott] *Every countable MV algebra is isomorphic to a* $\mathsf{T}(S)$ *where* S *is AF.*

Wehrung (2015) has generalized this result to *arbitrary* MV algebras.

Example The direct limit of $I_1 \to I_2 \to I_4 \to I_8 \to \ldots$ is the *CAR inverse monoid* whose associated MV algebra is that of the dyadic rationals in $[0,1]$.

Remark The partial commutative monoid $\mathsf{T}(S)$ has been studied by Kudyravtseva, Lawson, Lenz and Resende in relation to the existence of *invariant means* on Boolean inverse monoids and *abstract Banach-Tarski theory*.

> ” A localization may profitably be viewed as a non-commutative analog (sic) of a countable basis; its affiliated inverse semigroup is to be viewed as the analog (sic) of a topology.” Alexander Kumjian, 1984.

A *Boolean space* is a compact Hausdorff space with a basis of clopen subsets.

The following can be deduced from Corollary 2.

Theorem 9 [Lawson, Lawson & Lenz]

1. *Boolean inverse monoids are in duality with étale topological spaces with a Boolean space of identities.*

2. *(Countable) Boolean inverse $\wedge$-monoids are in duality with (second countable) Hausdorff étale topological spaces with a Boolean space of identities.*

Example There is a family of Boolean inverse $\wedge$-monoids C_n, where $n \geq 2$, called *Cuntz inverse monoids* which are congruence-free and whose groups of units are the Thompson groups V_n. Their associated groupoids are the ones derived from Cuntz C^*-algebras.

This is where the lecture as given ended due to mutual exhaustion on the part of audience and lecturer.

The ideas that follow were partly inspired by work of Matui.

- A Boolean inverse monoid is *basic* if each element is a join of a finite number of infinitesimals and idempotents.

- A groupoid is *principal* if it is derived from an equivalence relation.

- A topological groupoid G is *minimal* if every G-orbit is a dense subset of the space of identities.

- A topological groupoid is *effective* if $\mathrm{Iso}(G)^{\circ}$ is equal to the space of identities. Here $\mathrm{Iso}(G)$ is the union of the local groups.

Example The finite symmetric inverse monoids are basic; AF inverse monoids are basic.

Theorem 10 [Lawson]

Boolean inverse $\wedge$-monoid	*étale groupoid*
fundamental	*effective*
0-simplifying	*minimal*
basic	*principal*

We call the countable atomless Boolean algebra the *Tarski algebra*. Under Stone duality the Tarski algebra corresponds to the Cantor space.

A *Tarski inverse monoid* is a countable Boolean inverse $\wedge$-monoid whose set of idempotents forms a Tarski algebra.

Theorem 11 [Lawson] *There are bijective correspondences between the following three classes of structures.*

1. *Fundamental (*0*-simplifying) Tarski inverse monoids.*

2. *Second countable Hausdorff étale topological effective (minimal) groupoids with a Cantor space of identities.*

3. Cantor groups*: full countable (minimal) subgroups of the group of homeomorphisms of the Cantor space in which the support of each element is clopen.*

Theorem 12 [Lawson *after* Krieger] *There is a bijective correspondence between the following two classes of structures.*

1. *Basic locally finite factorizable Tarski inverse monoids (they are AF).*

2. Ample groups*: locally finite Cantor groups in which the fixed-point set of each element is clopen.*

A Boolean inverse monoid is said to be *piecewise factorizable* if each element s can be written $s = \bigvee_{i=1}^{n} s_i$ where each $s_i \leq g_i$, a unit.

Theorem 13 [Lawson] *A 0-simplifying Tarski inverse monoid is piecewise factorizable.*

4. Further questions

- Classify Tarski inverse monoids. Can non-classical symmetry phenomena be shown to arise in this way?

- Develop a Morita theory for Boolean inverse monoids.

- Study the classifying topos (respectively, space) of a Boolean inverse monoid.

- What is the ‘logic’ of Boolean inverse monoids?

- What is the nature of the relationship between inverse semigroups and C^*-algebras (and von Neumann algebras)?

- Develop the theory of coverings on inverse semigroups (not discussed here, but important in constructing Boolean inverse monoids in specific contexts, such as from tilings).