Distributive inverse semigroups

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This talk is based on work carried out in collaboration with Johannes Kellendonk, Ganna Kudryavtseva, Daniel Lenz, Stuart Margolis, and Ben Steinberg.

The main idea

The most succesful branch of semigroup theory has been *finite* semigroup theory.

Why?

Because there are close links between such semigroups and automata theory and the theory of regular languages.

These links have helped guide the theory, suggested problems and unified it.

Inverse semigroup theory has been successful but has apparently lacked the unifying guiding principles of finite semigroup theory.

My aim in this talk will be to propose just such a guiding principle.

Inverse semigroup theory can be traced back to the work of three mathematicians

- Ehresmann
- Wagner
- Preston

All three were motivated by the desire to algebraicize the concept of a pseudogroup of transformations.

In fact, both Ehresmann and Wagner were differential geometers.

Ehresmann, together with his student Bénabou, are also credited with the introduction of a lattice-theoretic approach to studying topological spaces called *frame theory*.

A *frame* is a complete infinitely distributive lattice.

The open subsets of a topological space form a frame.

$$\text{points} \longrightarrow \text{topological spaces}$$

$$\text{open sets} \longrightarrow \text{frames}$$

Johnstone's book *Stone spaces* develops this idea.

The key point is that Ehresmann arrived at the notion of a frame through his algebraicization of the notion of a pseudogroup.

Ehresmann's work can be phrased in terms of inverse semigroups.

Main idea

Inverse semigroup theory should be viewed as (part of) non-commutative frame theory. This approach provides natural connections with the theories of topoi, quantales and C^*-algebras.

The goal of this talk is to provide one illustration of this idea.

The main idea was motivated *in general* by a growing body of work in which inverse semigroups are used to construct C^*-algebras (Exel, Paterson, Renault, Resende etc)

It was motivated *in particular*

- by work of Kellendonk on tiling semigroups and topological groupoids carried out in 1997.

- by work of Lenz, motivated by the above, carried out in 2002 though only published in 2008.

- by work of Birget on constructing the Thompson groups from polycyclic inverse monoids published in 2004.

Topics

1. Definitions

2. Non-commutative Stone dualities

3. Examples

4. Constructing distributive inverse semigroups

5. The key example

1. Definitions

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

An inverse semigroup S is equipped with two important relations.

$s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

$s \sim t$ if and only if st^{-1} and $s^{-1}t$ both idempotents. Called the *compatibility relation*. It controls when pairs of elements are *eligible* to have a join.

Lattices need not have 1's but always have 0's. If they have 1's they will be called *unital*.

Thus: distributive lattices vs. unital distributive lattices; Boolean algebras vs. unital Boolean algebras.

A *distributive inverse semigroup* is one which has joins of compatible pairs of elements and multiplication distributes over such joins.

A *Boolean inverse semigroup* is a distributive inverse semigroup with a Boolean algebra of idempotents.

A *Boolean inverse* $\wedge$*-semigroup* is a Boolean inverse semigroup with the additional property that all pairs of elements have a meet.

A vanilla distributization.

Theorem [Schein] *Let S be an inverse semigroup. There is a distributive inverse semigroup* D(S) *and a map $\delta\colon S \to$* D(S) *which is universal for maps from S to distributive inverse semigroups.*

Let P be a poset with zero 0.

A subset $F \subseteq P$ is a *filter* if it is downwardly directed and upwardly closed.

It is *proper* if $0 \notin F$; all filters will be proper.

An *ultrafilter* is a maximal proper filter.

A filter F is *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

2. Non-commutative Stone dualities

A *groupoid* G is a (for us, small) category with every arrow invertible. The set of identities (or objects) of G is denoted by G_o. The 'o' stands for 'objects'.

If a groupoid G carries a topology making the multiplication and inversion continuous, it is called a *topological groupoid*.

The most important class of topological groupoids are the *étale groupoids*. We use Resende's characterization to define them.

A topological groupoid G is *étale* if G_o is an open set and the product of any two open sets in G is an open set.

N.B. Hausdorffness is not assumed.

A topological space is said to be *sober* if each point of the space is uniquely determined by the open sets that contain it (plus a bit more.)

A topological space X is said to be *spectral* if it is sober and has a basis of compact-open sets that is closed under finite non-empty intersections.

We do not assume that X is compact.

An étale groupoid is called *spectral* if its space of identities is a spectral space.

A étale groupoid is called *Boolean* if its space of identities is Boolean.

To avoid piling on definitions, *morphisms* will be kept in the background throughout this talk — they can be defined so that things work.

Classical theorems.

Theorem [Stone duality for distributive lattices] *The category of distributive lattices and their proper homomorphisms is dually equivalent to the category of spectral spaces and their coherent continuous maps.*

A Hausdorff spectral space is called a *Boolean space*.

Theorem [Stone duality for Boolean algebras] *The category of Boolean algebras and their proper homomorphisms is dually equivalent to the category of Boolean spaces and their coherent continuous maps.*

The starting point of our work.

Theorem [Stone duality for distributive inverse semigroups] *The category of distributive inverse semigroups is dually equivalent to the category of spectral groupoids.*

Theorem [Stone duality for Boolean inverse semigroups] *The category of Boolean inverse semigroups is dually equivalent to the category of Boolean groupoids.*

Theorem [Stone duality for Boolean inverse $\wedge$-semigroups] *The category of Boolean inverse $\wedge$-semigroups is dually equivalent to the category of Hausdorff Boolean groupoids.*

Proof sketch

Let G be a spectral groupoid.

A *local bisection* A of a groupoid G is a subset such that $A^{-1}A, AA^{-1} \subseteq G_o$. The set of all compact-open local bisections is a distributive inverse semigroup.

Let S be a distributive inverse semigroup.

Let P be a prime filter. Define $\mathbf{d}(P) = (P^{-1}P)^{\uparrow}$ and $\mathbf{r}(P) = (PP^{-1})^{\uparrow}$. Define the partial product $P \cdot Q$ to be $(PQ)^{\uparrow}$ iff $\mathbf{d}(P) = \mathbf{r}(P)$. In this way, the set of prime filters becomes a groupoid $\mathsf{G}_P(S)$.

Let $s \in S$. Define X_s to be the set of all prime filters that contain s. These sets form the basis of a topology on $\mathsf{G}_P(S)$.

Higher level proof

A *pseudogroup* is an inverse semigroup with arbitrary non-empty compatible joins and infinite distributivity.

There is an adjunction between the dual of the category of pseudogroups and the category of étale groupoids due to Resende (without morphisms) and Lawson/Lenz (with morphisms).

The duality for distributive inverse semigroups results from this duality by restricting using *coherence*.

We are therefore in the world of *non-commutative frame theory*.

3. Examples

Let G be a finite discrete groupoid. The set of all local bisections of G is a finite Boolean inverse $\wedge$-semigroup $I(G)$ and all finite inverse $\wedge$-semigroups are of this form.

Write $G = \bigsqcup_{i=1}^{m} G_i$ where the G_i are the connected components of G. Then

$$I(G) \cong I(G_1) \times \ldots \times I(G_m).$$

Let G be a finite connected discrete combinatorial groupoid and put $G_o = X$. Then $I(G) \cong I(X)$, a finite symmetric inverse monoid.

The fundamental finite Boolean inverse $\wedge$-semigroups are therefore of the form

$$I(X_1) \times \ldots \times I(X_m).$$

Call these *semisimple*.

May construct *AF inverse monoids* from Bratteli diagrams and injective morphisms between semisimple inverse monoids.

4. Constructing distributive inverse semigroups

Let S be an inverse semigroup. Let $a \in S$ and $b_1, \ldots, b_m \leq a$. We say that the set of elements $\{b_1, \ldots, b_m\}$ is a *(tight) cover* of a if for each $0 \neq x \leq a$ there exists b_i such that $0 \neq z \leq x, b_i$ for some z.

A *tight filter* is a filter A such that if $a \in A$ and $\{b_1, \ldots, b_m\}$ covers a then $b_i \in A$ for some i.

A semigroup homomorphism $\theta: S \to T$ to a distributive inverse semigroup is said to be a *tight map* if for each element $a \in S$ and tight cover $\{a_1, \ldots, a_n\}$ of a we have that $\theta(a) = \bigvee_{i=1}^{n} \theta(a_i)$.

Intuitive idea

The idea is to present distributive inverse semigroups by means of *generators and relations*.

The generating set is in fact an inverse semigroup S.

The relations are given by the tight covers —

if $\{b_1, \ldots, b_m\}$ is a *(tight) cover* of a, then THINK

$$a = \bigvee_{i=1}^{m} b_i.$$

Theorem [Tight completions] *Let S be an inverse semigroup.*

1. *There is a distributive inverse semigroup $\mathsf{D}_t(S)$ and a tight map $\delta\colon S \to \mathsf{D}_t(S)$ which is universal for tight maps from S to distributive inverse semigroups.*

2. *There is an order isomorphism between the poset of tight filters in S and the poset of prime filters in $\mathsf{D}_t(S)$ under which ultrafilters correspond to ultrafilters.*

We call the distributive inverse semigroup $\mathsf{D}_t(S)$ the *tight completion* of S.

If the tight completion of an inverse semigroup is actually Boolean we say that the semigroup is *pre-Boolean*.

It can be proved that every ultrafilter is a tight filters.

Theorem *An inverse semigroup is pre-Boolean if and only if every tight filter is an ultrafilter.*

5. The key example

The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero generated by the variables $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0, i \neq j.$$

Every non-zero element of P_n is of the form yx^{-1} where x and y are elements of the *free monoid* on $\{a_1, \ldots, a_n\}$.

The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some } z \end{cases}$$

The polycyclic monoids are interesting in themselves.

But I now argue that they become even more interesting when viewed as generating an associated Boolean inverse monoid.

The polycyclic monoid P_n is a pre-Boolean inverse monoid.

The set $\{a_1 a_1^{-1}, \ldots, a_n a_n^{-1}\}$ is a tight cover of the identity, and in some sense, determines all other tight covers.

Theorem The Boolean completion of P_n is called (here) the *Cuntz inverse monoid* CI_n.

1. This monoid is congruence-free.

2. Its group of units is the Thompson group $V_{n,1}$.

3. Its associated groupoid is the groupoid also associated with the Cuntz C^*-algebra C_n.

O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, (1999) is, in fact, a study of tight maps from P_n to $I(X)$.

- All Thompson-Higman groups $V_{n,r}$ can be constructed in a similar way.

- Self-similar groups actions give rise to generalizations of the polycyclic inverse monoids which are also pre-Boolean.

- Finite directed graphs can be used to construct pre-Boolean inverse semigroups.

- AF inverse monoids are generated by pre-Boolean inverse monoids.

Our theory can be used to construct interesting groups of the Thompson-Higman variety.

Intuively, the elements of the group are obtained by *glueing together partial bijections*.

Thus our theory can be used to construct interesting groups from inverse semigroups.