Non-commutative Stone dualities

Mark V Lawson
Heriot-Watt University
and the
Maxwell Institute for Mathematical Sciences
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This talk is based on joint work carried out in collaboration with Ganna Kudryavtseva.

The big idea

Generalize classical Stone duality to a non-commutative setting using semigroups and quantales.

Regard this theory as *non-commutative frame theory*.

Explore applications to C^*-algebras, group theory, tilings, . . .

0. Stone duality

The following is a classical theorem due to Marshall H. Stone.

Theorem *The category of unital Boolean algebras is dual to the category of* Boolean spaces *— that is, compact Hausdorff topological spaces with a basis of clopen sets.*

In general terms, this theorem links algebra, in the guise of Boolean algebras, with topology.

The Boolean algebras should be regarded as *commutative* structures.

The aim of this talk is to show you how this theorem may be generalized in such a way that the *commutative* algebraic structures are replaced by *non-commutative* structures.

1. Motivations

The following are all related and provided important examples and motivations.

- The work of Ehresmann on ordered categories from the 1950s.

- The work of Renault from the 1980s and Paterson from the 1990s linking inverse semigroups, étale topological groupoids and C^*-algebras.

- The work of Kellendonk on tiling semigroups and topological groupoids carried out in 1997.

- The work of Lenz, motivated by the above, carried out in 2002 though only published in 2008.

- The work of Birget on constructing the Thompson groups from polycyclic inverse monoids published in 2004.

- The theory of frames and locales.

- **The work of Resende in 2007 in which inverse semigroups and localic groupoids are linked by quantales**.

2. First example

We shall replace Boolean algebras by *monoids* with extra structure. To explain what that extra structure is we shall examine a concrete example in detail. This will motivate the whole talk.

Let X be a non-empty set. Denote by B(X) the set of all binary relations on X. Equip this set with the usual multiplication of binary relations. This turns B(X) into a monoid with zero 0, the empty relation.

[More generally, replace B(X) by B(A) where A is any reflexive and transitive relation on X.]

There is all kinds of extra structure here. We have to decide what is relevant to our main aim.

Resende's paper provided the main clue.

Denote by E the set of all binary relations which are *partial identities*. That is, binary relations a such that $(y, x) \in a$ implies that $y = x$. These are idempotents in $\mathsf{B}(X)$ that we call *projections*. These form a commutative idempotent submonoid.

Let a be an arbitrary binary relation. Define

$$\lambda(a) = \{(x, x) \in X \times X \colon \exists y, (y, x) \in a\}$$

and

$$\rho(a) = \{(y, y) \in X \times X \colon \exists x, (y, x) \in a\}.$$

Both of these elements are projections.

We have a monoid S, a submonoid of idempotents E which is commutative and two maps $\lambda, \rho \colon S \to E$ in which the following axioms are satisfied.

(ES1) If $a \in E$ then $\lambda(a) = a = \rho(a)$.

(ES2) $a\lambda(a) = a = \rho(a)a$.

(ES3) $\lambda(\lambda(a)b) = \lambda(ab)$ and $\rho(a\rho(b)) = \rho(ab)$.

Such a monoid is called an *Ehresmann monoid* (w.r.t the set of *projections* E.)

Every Ehresmann monoid is equipped with a *natural partial order* $\preceq$ defined by

$$a \preceq b \Leftrightarrow a = eb = bf$$

where e and f are projections.

Not necessarily compatible with the multiplication.

This will play an important role later.

A *quantale* is a (sup-)lattice-ordered semigroup in which multiplication distributes over joins. Our quantales will be *unital*. The identity denoted by e.

A *frame* is a sup-lattice in which finite meets distribute over arbitrary joins.

A *unital quantal frame* is a unital quantale which is also a frame.

An *Ehresmann quantal frame* is a unital quantal frame that is an Ehresmann monoid with respect to the set of projections $e^{\downarrow}$ and in which λ and ρ are sup-maps.

Proposition *For any reflexive and transitive relation A on a set X, the monoid* $\mathsf{B}(A)$ *is an Ehresman quantal frame.*

Ehresmann quantal frames satisfying an additional property to be described later will be the main algebraic objects we shall study.

3. Second example

The monoid $\mathsf{B}(X)$ is the set of all subsets of $X \times X$.

We can regard $X \times X$ as a category, in fact a groupoid with the discrete topology.

More generally, let C be a topological category in which the maps $\mathbf{d}$, $\mathbf{r}$ and $\mathbf{m}$ are all open and where the space of identities C_o is an open subset.

Denote by $\mathsf{O}(C)$ the set of all open subsets of C. Denote by E the set of all open subsets of C_o. For $A \in \mathsf{O}(C)$ define

$\lambda(A) = \{\mathbf{d}(a) \colon a \in A\}$ and $\rho(A) = \{\mathbf{r}(a) \colon a \in A\}$.

Since both $\mathbf{d}$ and $\mathbf{r}$ are open maps, we have well-defined maps

$$\lambda, \rho \colon \mathsf{O}(C) \to E.$$

If $A, B \in \mathsf{O}(C)$ define AB to be the binary operation $\mathsf{O}(C)$, well-defined because $\mathbf{m}$ is open.

Proposition *With the above definitions,* $\mathsf{O}(C)$ *is an Ehresmann quantal frame.*

Our results so far hint at a correspondence linking

- Ehresmann quantal frames.
- Certain topological categories.

It is more covenient to work with another formulation of what we mean by a 'space'.

4. Frames and locales

We follow Resende and work in the first instance not with topological spaces but with locales.

A *frame* is a complete lattice satisfying the infinite distributive law. A *morphism* between frames is a map preserving finite meets and arbitrary joins.

The opposite of the category of frames is called the category of *locales*.

With each topological space X, we may associate its frame $\Omega(X)$ of open subsets.

With each locale A we may associate the space pt(A) whose points are the completely prime filters on A. The open sets are those sets of the form V_a, where $a \in A$, and V_a consists of all completely prime filters containing a.

The following is a standard result.

Theorem

1. *There is an adjunction* $\Omega \dashv$ *pt between the category of spaces and the category of frames.*

2. *This adjunction restricts to an equivalence between* sober *spaces and* spatial *locales.*

We shall accordingly work with *localic categories* rather than topological categories.

This consists of two locales C_1 and C_0 where C_1 should be regarded as the arrows and C_0 as the identities. These are connected by locale maps

$$u: C_0 \to C_1, \quad d, r: C_1 \to C_0, \quad m: C_1 \times_{C_0} C_1 \to C_1$$

that satisfy the obvious conditions to give us a category.

A locale map is said to be *semiopen* if its associated frame map preserves arbitrary meets.

A locale map is said to be *open* if it is semiopen and satisfies an algebraic condition called the *Frobenius reciprocity condition*.

4. Theorem

A *quantal localic category* is a localic category where the maps d, r, u are open and m is semiopen.

An Ehresmann quantal frame Q is said to be *multiplicative* if the multiplication map

$$Q \otimes_{e\downarrow} Q \to Q$$

has a right adjoint that preserves arbitrary joins.

We omit describing the morphisms that make sense of the following statement.

The correspondence theorem *There is a duality between multiplicative Ehresmann quantal frames and quantal localic categories.*

This theorem provides a common framework for

- Resende's work on the relationship between étale groupoids, quantales, and inverse semigroups.

- Work by Renault, Paterson, Kellendonk, Lenz, Exel etc on the role of of inverse semigroups in the theory of C^*-algebras.

We shall now show that there is also a connection with the York school of semigroup theory.

5. Restriction monoids

Multiplicative Ehresmann quantal frames seem like rather abstract structures. We now show how to construct an interesting class of examples and, in the process, refine the statement of our main theorem.

An Ehresmann semigroup is said to be a *restriction semigroup* if it satisfies the following two axioms:

(RS1) $ea = a\lambda(ea)$ for all projections e.

(RS2) $af = \rho(af)a$ for all projections f.

On a restriction semigroup the natural partial order is compatible with the multiplication.

Elements a and b are said to be *compatible*, denoted $a \sim b$ if and only if $a\lambda(b) = b\lambda(a)$ and $\rho(b)a = \rho(a)b$.

A *complete restriction monoid* is a restriction monoid whose projections form a frame, all joins of compatible subsets exist and products ditribute over the joins that exist.

Let Q be an Ehresmann quantale.

This is equipped with two orders $\leq$ and $\preceq$.

Observe that

$$a \preceq b \Rightarrow a \leq b.$$

We use the relationship between these two orders to define an important class of elements.

An element $a \in Q$ is called a *partial isometry* if for all $b \in Q$ we have that

$$b \leq a \Rightarrow b \preceq a.$$

We denote the set of partial isometries of Q by PI(Q). This is not always a submonoid.

Example Let $X = \{1, 2\}$ and let A be the reflexive and transitive relation $\leq$. The set of partial isometries of B(A) is $I(X) \setminus \{(2, 1)\}$. In particular, it need not be an inverse monoid.

Let S be a complete restriction monoid. Denote by $\mathsf{L}^{\vee}(S)$ the set of all order ideals of S closed under compatible joins.

Theorem $\mathsf{L}^{\vee}(S)$ *is an Ehresmann quantal frame. In addition,*

1. *The partial isometries of* $\mathsf{L}^{\vee}(S)$ *form a submonoid isomorphic to* S*.*
2. *Each element of* $\mathsf{L}^{\vee}(S)$ *is a join of partial isometries.*
3. $\mathsf{L}^{\vee}(S)$ *is multiplicative.*

A *restriction quantal frame* is an Ehresmann quantal frame Q in which the top element of Q is a join of partial isometries and the partial isometries form a submonoid.

An *étale localic category* is a localic category in which the maps u and m are open and d and r are étale.

Theorem *The category of complete restriction monoids is equivalent to the category of restriction quantal frames.*

Main theorem *The category of complete restriction monoids is dual to the category of étale localic categories.*

By incorporating involutions into the above theory, we may deduce the main theorem of Resende (2007). Recall that a *pseudogroup* is a complete inverse monoid.

Theorem *The category of pseudogroups is dual to the category of étale localic groupoids.*

We may replace localic categories by topological but we pay the usual price.

Theorem *The category of spatial complete restriction monoids is dually equivalent to the category of sober étale topological groupoids.*

6. Inverse monoids

A *distributive inverse semigroup* is one with finite non-empty compatible joins and multiplication distributes over finite joins.

A *Boolean inverse semigroup* is a distributive inverse semigroup in which the idempotents form a Boolean algebra.

An *inverse* $\wedge$*-semigroup* is an inverse semigroup that has all binary meets.

A *spectral space* is a sober space that has a basis of compact-open subsets closed under finite non-empty intersections.

A *Boolean space* is a Hausdorff spectral space.

A *spectral groupoid* is an étale groupoid with a spectral space of identities.

A *Boolean groupoid* is an étale groupoid with a Boolean space of identities.

From the theorem above, we may deduce the following which are important in relating inverse semigroups to C^*-algebras.

Theorem

1. *The category of distributive inverse semigroups is dually equivalent to the category of spectral groupoids.*

2. *The category of Boolean inverse semigroups is dually equivalent to the category of Boolean groupoids.*

3. *The category of Boolean inverse $\wedge$-semigroups is dually equivalent to the category of Hausdorff Boolean groupoids.*