Non-commutative Stone dualities and étale groupoids

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Algebra	**Topology**
Semigroup	Locally compact
Monoid	Compact

For simplicity, in this talk my focus will be on monoids and so compact spaces.

1. Stone, 1937

In 1937, Marshall Stone wrote a paper

M. H. Stone, Applications of the theory of Boolean rings to general topology, *Transactions of the American Mathematical Society* **41** (1937), 375–481.

in which he generalized the theory of *finite* Boolean algebras to *arbitrary* Boolean algebras.

This theory is now known as Stone duality.

Stone was what we would now term a functional analyst.

Question: How did he become interested in Boolean algebras?

Answer: Through algebras of commuting projections.

More generally . . .

Let R be a commutative ring.

Denote by $\mathsf{E}(R)$ the set of idempotents of R.

On the set $\mathsf{E}(R)$ define

$$a \wedge b = a \cdot b \quad a \vee b = a + b - ab \quad a' = 1 - a.$$

Theorem $(\mathsf{E}(R), \wedge, \vee, {}', 0, 1)$ *is a Boolean algebra and every Boolean algebra arises in this way.*

2. Stone duality

Stone's work is the first deep result on Boolean algebras.

It is also represents the first construction of a topological space from *algebraic data.*

Define a *Boolean space* to be a 0-dimensional, compact Hausdorff space.

Theorem [Stone, 1937]

1. *Let S be a Boolean space. Then the set* $\mathsf{B}(S)$ *of clopen subsets of S is a Boolean algebra.*

2. *Let A be a Boolean algebra. Then the set* $\mathsf{X}(A)$ *of all ultrafilters of A can be topologized in such a way that it becomes a Boolean space. It is called the* Stone space *of A.*

3. *If S is a Boolean space then $S \cong \mathsf{XB}(S)$.*

4. *If A is a Boolean algebra then $A \cong \mathsf{BX}(A)$.*

Examples

1. Up to isomorphism, there is exactly one countable, atomless Boolean algebra. It is innominate so I call it the *Tarski algebra*. The Stone space of the Tarski algebra is the *Cantor space.*

2. The Stone space of the powerset Boolean algebra P(X) is the Stone-Čech compactification of the discrete space X.

3. Cuntz, 1977

In his paper

J. Cuntz, Simple C^*-algebras generated by isometries, *Communications in Mathematical Physics* **57**, (1977), 173–185.

Cuntz constructs a family of separable, simple infinite C^*-algebras, $\mathcal{O}_n$, called today the *Cuntz algebras*.

$\mathcal{O}_n$ is generated by n isometries $S_1, \ldots, S_n$ such that $S_i^* S_i = 1$ and $\sum_{i=1}^{n} S_i S_i^* = 1$.

But the $S_1, \ldots, S_n$ generate an inverse submonoid of $\mathcal{O}_n$, called the *polycyclic monoid* P_n on n-generators. [This monoid arises in the study of pushdown automata and CF languages].

Inverse semigroups

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

An inverse semigroup S is equipped with two important relations:

1. $s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

2. $s \sim t$ if and only if st^{-1} and $s^{-1}t$ both idempotents. Called the *compatibility relation*. It controls when pairs of elements are *eligible* to have a join.

Example The symmetric inverse monoids $I(X)$ are the prototypes of inverse semigroups just as the symmetric groups are the prototypes of groups.

If X has n elements we sometimes denote the symmetric inverse monoid on n letters by I_n.

The idempotents of an inverse semigroup form a commutative subsemigroup but are not (ring theorists beware!) central.

4. Renault, 1980

Renault's monograph

J. Renault, *A groupoid approach to C^*-algebras*, LNM 793, Springer-Verlag, 1980.

highlighted the important role played by inverse semigroups in the theory of C^*-algebras.

There were earlier papers on the interactions between inverse semigroups and functional analysis:

B. A. Barnes, Representations of the l_1-algebra of an inverse semigroup, *Trans. Amer. Math. Soc.* **218** (1976), 361–396.

But since Renault's book, inverse semigroups have become a feature of the theory of C^*-algebras.

The work of Ruy Exel is particularly noteworthy

`http://mtm.ufsc.br/~exel/`.

Question: Why inverse semigroups and C^*-algebras?

Answer: Because the set of partial isometries of a C^*-algebra is *almost* an inverse semigroup.

The following is Theorem 4.2.3 of my book on inverse semigroups.

Theorem *The set of partial isometries of a C^*-algebra forms an ordered groupoid*

5. Boolean inverse monoids

Inverse semigroups might not, however, be the right structures to study in this context.

A *Boolean inverse monoid* is an inverse monoid satisfying the following conditions:

1. The set of idempotents forms a Boolean algebra under the natural partial order.
2. Compatible pairs of elements have a join.
3. Multiplication distributes over the compatible joins in (2).

Symmetric inverse monoids are Boolean.

The compatible joins give rise to a (partially) *additive structure.*

Theorem [Paterson 1999, Wehrung, 2017] *Let S be an inverse submonoid of the multiplicative monoid of a C^*-algebra R where $s^{-1} = s^*$ for each $s \in S$. Then there is a Boolean inverse monoid B such that $S \subseteq B \subseteq R$.*

Example Let S be the monoid that consists of the matrix units in $R = M_n(\mathbb{C})$ together with the zero and the identity. Then B is the Boolean inverse monoid of *rook matrices* in R. The monoid B is isomorphic to the symmetric inverse monoid on n letters.

- We view Boolean inverse monoids as non-commutative generalizations of Boolean algebras.

- Boolean inverse monoids are 'ring-like' with the partial join operation being analogous to the addition in a ring. Wehrung (2017) proved they form a variety and have a Mal'cev term.

- **This raises the question of generalizing Stone duality to a non-commutative setting.**

What, then, are the generalizations of Boolean spaces?

6. Etale groupoids

We shall regard groupoids as algebraic structures with a subset of *identities*. If G is a groupoid, its set of identities if G_o.

Examples

1. Groups are the groupoids with exactly one identity.

2. Equivalence relations can be regarded as principal groupoids; the *pair groupoid* $X \times X$ is a special case.

3. From a group action $G \times X \to X$ we get the *transformation groupoid* $G \ltimes X$.

A *topological groupoid* is a groupoid G equipped with a topological structure in which both multiplication and inversion are continuous.

A topological groupoid is said to be *étale* if the domain map is a local homeomorphism.

WHY ETALE?

If X is a topological space, denote by $\Omega(X)$ the lattice of all open sets of X.

Theorem [Resende, 2006] *Let G be a topological groupoid. Then G is étale if and only if $\Omega(G)$ is a monoid.*

- Etale groupoids are topological groupoids with an algebraic alter ego.
- Etale groupoids should be viewed as generalized spaces (Kumjian, Crainic and Moerdijk . . . .)

7. Non-commutative Stone duality

A *Boolean groupoid* is an étale groupoid whose space of identities is a Boolean space.

Let G be a groupoid. A *partial bisection* is a subset $A \subseteq G$ such that $A^{-1}A, AA^{-1} \subseteq G_o$.

Let G be a Boolean groupoid. The set of compact-open partial bisections of G is denoted by $\mathsf{B}(G)$.

Let S be a Boolean inverse monoid. The set of ultrafilters of S is denoted by $\mathsf{G}(S)$.

Theorem [Lawson & Lenz, Resende]

1. *Let G be a Boolean groupoid. Then* $\mathsf{B}(G)$ *is a Boolean inverse monoid.*

2. *Let S be a Boolean inverse monoid. Then* $\mathsf{G}(S)$ *is a Boolean groupoid, called the* Stone groupoid *of S.*

3. *If G is a Boolean groupoid then $G \cong \mathsf{GB}(G)$.*

4. *If S is a Boolean inverse monoid then $S \cong \mathsf{BG}(S)$.*

Example

An inverse semigroup is *fundamental* if the only elements centralizing the idempotents are idempotents. A Boolean inverse monoid is *simple* if it has no non-trivial *additive* ideals.

Theorem

1. The finite, fundamental Boolean inverse monoids are finite direct products
$$I_{n_1} \times \ldots \times I_{n_r}.$$
[Compare finite dimensional C^*-algebras.]

2. The finite simple Boolean inverse monoids are the finite symmetric inverse monoids $I(X)$.

3. The Boolean groupoid associated with $I(X)$ is the pair groupoid $X \times X$.

8. Envoi

The results of this talk can be placed in a broader setting.

Commutative	**Non-commutative**
Meet semilattice	Inverse semigroup
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup

There are also connections with topos theory, quantale theory, infinite simple groups, MV-algebras, aperiodic tilings . . .

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