Monoids, groups and groupoids

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November 2nd 2016

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Background

This work originated in my attempt to understand a paper by Birget on constructing the classical Thompson groups from semigroups. As my work progressed, it became clear that there were connections with earlier papers by Kellendonk which examined the theory of aperiodic tilings from the perspective of groupoid C^*-algebras. Lenz's algebraic analysis of Kellendonk's work combined with papers within C^*-algebra theory founded on Renault's groupoid approach led to the starting point of my research: <u>that we were dealing with a non-commutative generalization of classical Stone duality</u>.

This viewpoint has been enormously fruitful. In particular, it has led to new ~~work~~ coordinatization of MV-algebras, an abstract ~~theory~~ ~~providing~~ a setting for the Banach-Tarski paradox, and close connections with the theory of "topological full groups".

In this talk, however, I shall focus on setting up the duality theory and touching on the nature of the groups that arise.

The big picture

~~Non-commutative~~

Boolean inverse monoids

duality

Etale groupoids with identity spaces being Boolean spaces

Non-commutative

Commutative

Boolean algebras

duality

Boolean spaces

(= Compact, 0-dim'l Hausdorff spaces)

~~Non-commutative~~

Tarski algebra

Cantor space

Boolean inverse monoids

- inverse monoids
- all compatible pairs of elements have a join
- multiplication distributes over such binary joins.
- set of idempotents form a Boolean algebra

It is called a <u>$\wedge$-monoid</u> if all pairs of elements have meets.

Etale groupoids

A topological groupoid in which the domain (and range) maps are local homeomorphisms.

Significance?

Theorem (Resende) A topological groupoid G is étale if and only if its set of open subsets, $\Omega(G)$, is a monoid under multiplication of subsets.

Thus, étale groupoids have an algebraic character.

A Boolean groupoid is an étale groupoid whose space of identities is a Boolean space.

Functors (Contravariant, as it happens, but I will omit morphisms)

• Let G be a Boolean groupoid. Define $KB(~~B~~G)$ to be the set of all compact-open local bisections of G.

[$A \subseteq G$ is called a local bisection if $A^{-1}A$, AA^{-1} are both sets of identities]

Then $KB(G)$ is a Boolean inverse monoid

• Let S be a Boolean inverse monoid. Denote by $\underline{G}(S)$ the set of all <u>ultrafilters</u> on S.

[~~B~~ $F \subseteq S$ is called an ultrafilter if

· $0 \notin F$; $a, b \in F \Rightarrow \exists c \in F$, $c \leq a, b$; $a \in F$ and $a \leq b \Rightarrow b \in F$; F is maximal with these properties]

Then $\underline{G}(S)$ can be endowed with the structure of a Boolean groupoid.

[Idea: ultrafilters "look like" cosets of ultrafilters in the Boolean algebra $E(S)$].

Theorem (Non-commutative Stone duality). The category of Boolean inverse monoids is dually equivalent to the category of Boolean groupoids.

Under this duality, Hausdorff Boolean groupoids correspond to Boolean inverse $\wedge$-monoids

Examples

1. <u>Finite symmetric inverse monoids</u>
(groups of units the finite symmetric groups).

2. The <u>Cuntz inverse monoids</u> C_n $(n \geqslant 2)$
discrete analogues of the Cuntz C^*-algebras
(groups of units the Thompson groups V_n)

3. The <u>AF inverse monoids</u> discrete analogues of the AF C^*-algebras
(groups of units of direct limits of products of finite symmetric groups via diagonal embeddings).

Simple, Boolean inverse monoids

To get a handle on Boolean inverse monoids, we adopt the usual ploy in algebra of starting with the "simple" algebras.

- A Boolean inverse monoid is fundamental if the centralizer of its set of idempotents is just the set of idempotents.
- A Boolean inverse monoid is 0-simplifying if the only $\vee$-closed order ideals are the trivial ones.

$$\boxed{\text{Simple} = \text{fundamental} + \text{0-simplifying}}$$

The Tarski algebra is the (unique) countable atomless Boolean algebra. Under classical Stone duality it corresponds to the Cantor space

Theorem Let S be a simple countable Boolean inverse monoid.

Then

(1) If S is finite then S is isomorphic to a finite symmetric inverse monoid.

(2) If S is infinite then its Boolean algebra of idempotents is the Tarski algebra.

Idea

Thus the groups of units in (2) should be viewed as generalizations of finite symmetric groups.

<u>Theorem</u> The groups of units of countably infinite simple Boolean inverse monoids are precisely the subgroups of the group of homeos of the Cantor space which are minimal (in the topological sense) and "full" (= closure under a notion of "glueing").

If S is a monoid then $U(S)$ denotes its group of units.

<u>Theorem</u> (Matui – an algebraic proof by me is in preparation). Let S and T be two simple, countably infinite Boolean inverse $\wedge$-monoids. Then the following are equivalent:

(1) $S \cong T$.

(2) $U(S) \cong U(T)$.

Remark Let S be a simple, countably infinite Boolean inverse $\wedge$-monoid. Then in fact

$$S = \left(\mathsf{U}(S)^{\downarrow} \right)^{\vee}.$$

My proof is an algebraic version of Matui's and might ~~[illegible]~~ have an application in characterizing abstractly the groups of units that arise.

Thus (inverse) monoids, groups ~~[illegible]~~ and groupoids contain the same information in different incarnations

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→ Connections with C^*-algebras (Exel, ...)

→ Connections with the important ongoing work of Matui.

→ Connections with the "groups of dynamical origin" of Nekrashevych

→ Connections with work of Fred Wehrung (Caen).

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