Étale groupoids and their morphisms

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Skew Boolean algebras

A *lattice* is an algebra $(B, \wedge, \vee)$, where the binary operations $\wedge$ and $\vee$ are commutative, associaltive and satisfy absorption identities.

A *Boolean algebra* is a distributive lattice with the bottom element 0, and for any elements a, b there is $c = a \setminus b$ such that $(a \wedge b) \wedge c = 0$ and $(a \wedge b) \vee c = a$.

A *skew lattice* is an algebra $(B, \circ, \bullet)$, where the binary operations $\circ$ and $\bullet$ are associaltive and satisfy absorption identities (*commutativity not required!*).

A *skew Boolean algebra* is a skew lattice B with the bottom element 0 and in addition

- it is symmetric: $x \circ y = y \circ x \Leftrightarrow x \bullet y = y \bullet x$ and
- $x^{\downarrow} = \{x \circ y \circ x : y \in B\}$ is a Boolean algebra for any $x \in B$.

Skew Boolean algebras: some history

- First works on skew Boolean algebras are due to W. Cornish (1980) and J. Leech (1989).
- The study of non-commutative lattices was initiated by P. Jordan in 1949 motivated by a connection of non-commutative logic and quantum mechanics.
- Different authors have studied algebras $(B, \circ, \bullet)$ where $(B, \circ)$ and $(B, \bullet)$ are two bands, connected by some absorption rules.
- Instead of $\circ$ and $\bullet$ some authors considered quasiorders (Gerhards, Schein, Schweirert).

Skew Boolean algebras: structure

Some properties

- $(B, \circ)$ is a normal band and $(B, \bullet)$ is a regular band.
- The natural order on $(B, \bullet)$ is opposite to the natural order on $(B, \circ)$.
- The relation $\mathcal{L}$ on $(B, \bullet)$ equals the relation $\mathcal{R}$ on $(B, \circ)$ and dually.
- The relation $\mathcal{D}$ is the same on $(B, \bullet)$ and $(B, \circ)$.
- $B/\mathcal{D}$ is the maximal Boolean algebra quotient of B.

Fibered product decomposition

Define the order and the relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on B as those on $(B, \circ)$.
Then $B \simeq B/\mathcal{R} \times_{B/\mathcal{D}} B/\mathcal{L}$.

- $B/\mathcal{R}$ is left-handed: $(B, \circ)$ is a left normal and $(B, \bullet)$ a right regular band.
- $B/\mathcal{L}$ is right-handed.

Skew Boolean algebras: alternative view

Let $(B, \circ)$ be a normal band and let $\gamma : B \to B/\mathcal{D}$ be the canonical morphism. Since $(B, \circ)$ is a strong semilattice of rectangular bands, for any $a \in B$ and $D \in B/\mathcal{D}$ with $\gamma(A) \geq D$ the restriction $a|_D$ is well defined.

Compatibility relation

$a, b \in B$ are called compatible, if $a|_{\gamma(a)\wedge\gamma(b)} = b|_{\gamma(a)\wedge\gamma(b)}$.

Definition

A normal band $(B, \circ)$ is called Boolean provided that it has the bottom element, the semilattice $B/\mathcal{D}$ admits the structure of a Boolean algebra and moreover joins of compatible pairs of elements exist.

Observation (GK and M.V. Lawson)

The category of (right) Boolean normal bands is isomorphic to the category of (right-handed) skew Boolean algebras.

Connection with sheaves: outline

- Right normal bands are strong semilattices of right zero semigroups and can be looked at just as presheaves of sets over semilattices.
- Boolean right normal bands can be looked at as sheaves over Boolean algebras considered as poset-categories and endowed with a Grothendieck topology. This leads to an equivalence of categories.
- By the classical Stone duality, Boolean algebras are dual to Boolean spaces, and then Boolean right normal bands are dual to sheaves with global support, equivalently to étale spaces, over Boolean spaces.

Some references

1. GK, A refinement of Stone duality to skew Boolean algebras, *Algebra Universalis*, **67** (2012), 397–416.
2. GK, A dualizing object approach to non-commutative Stone duality, *J. Aust. Math. Soc.*, to appear.
3. A. Bauer, K. Cvetko-Vah, Stone duality for skew Boolean algebras with intersections, *Houston J. Math.*, to appear.
4. A. Bauer, K. Cvetko-Vah, M. Gehrke, S. van Gool, GK, A non-commutative Priestley duality, preprint.
5. GK, M.V. Lawson, Boolean sets, skew Boolean algebras and a non-commutative Stone duality, preprint.

Boolean right normal bands: going to points

B — Boolean right normal band. Let $\gamma : B \to B/\mathcal{D}$ be the canonical morphism.

- Let G be an ultrafilter of B. There is a unique ultrafilter F of $B/\mathcal{D}$ such that $\gamma(G) = F$ and for some (equiv. for any) $a \in G$

$$G = G_{a,F} = \{b \in B \colon b \Cap_F a\} = \{b \in B : \text{ there is } c \in B \text{ with } \gamma(c) \in F \text{ and } a, b \geq c\}.$$

- Ultrafilters correspond to morphisms $B \to \{0, 1, 2\}$ (with $\mathcal{D}$-classes: $\{1, 2\}$ with right zero multiplication and $\{0\}$) such that the inverse image of 1 is non-empty and minimal.
- We obtain the étale space $(\mathcal{U}(B), p, \mathcal{U}(B/\mathcal{D}))$.
- Elements of B can be recovered as sections of $\mathcal{U}(B)$ over compact-open sets. The operation on sections is $s \circ t = t|_{p(s) \wedge p(t)}$.

Frames and locales

A sup-lattice is a lattice with all (empty, finitaty, infinite) joins. A frame is a sup-lattice that is infinitely distributive:

$$a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge b_i).$$

Frame morphisms, by definition, preserve any joins and finite meets. The category of locales is defined to be opposite to the category of frames.

Various commutative dualities

- Frames are opposite to locales.
- Spatial frames are dual to sober spaces.
- Coherent frames are dual to coherent spaces.
- Distributive lattices are dual to coherent spaces.
- Boolean algebras are dual to Boolean spaces.

An extension to a non-commutative setting

In the commutative dualities, listed on the previous slide, we replace:

- Frame $\to$ right normal band closed under compatible joins over a frame.
- Locale $\to$ sheaf over a locale, space $\to$ sheaf ($=$ étale space) over a space.
- Morphism of frames $\to$ morphism of right normal bands over a frame morphism.
- Locale map (space map) $\to$ natural transformations of sheaves:
  - $\mathcal{A}$ and $\mathcal{B}$ — sheaves over locales X and Y, respectively.
  - $f : X \to Y$ — a map of locales.
  - $f_*(\mathcal{A})$ — the direct image sheaf of $\mathcal{A}$.

  Both $\mathcal{B}$ and $f_*(\mathcal{A})$ are functors from open sets of Y to $Sets^{op}$. Then a morphism is just a natural transformations of functors

$$\mathcal{B} \mapsto f_*(\mathcal{A}).$$

Another non-commutative setting: pseudogroups

Let S be an inverse semigroup and $a, b \in S$. We say that a, b are compatible provided that ab^{-1} and $a^{-1}b$ are idempotents.
A *pseudogroup* is an inverse semigroup S whose idempotents form a frame and such that joins of any compatible family of elements exist in S.

Examples

- Pseudogroups of homeomorphisms between opens of a topological space.
- Symmetric inverse semigroup $\mathcal{I}(X)$.
- Pseudogroup of open bisections of a topological groupoid with open multiplication map.

Some results and references

Equivalence between pseudogroups and inverse quantal frames. Duality between inverse quantal frames and localic etale groupoids, at the level of objects.

- P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.
- P. Resende; Lectures on étale groupoids, inverse semigroups and quantales, lecture notes, 2006, preprint.

Duality betwen spatial pseudogroups and sober étale groupoids, both at the levels of objects and morphisms (meet-preserving).

- M.V. Lawson, D. H. Lenz, Pseudogroups and their étale groupoids, 2011, preprint.
- M.V. Lawson, D. H. Lenz, Distributive inverse semigroups and non-commutative Stone dualities, 2013, preprint.
- M. V. Lawson, A non-commutative generalization of Stone duality, *J. Aust. Math. Soc.* **88** (2010), 385–404.
- M. V. Lawson, Non-commutative Stone duality: inverse semigroups, topological groupoids and $C^{\star}$-algebras, *Internat. J. Alg. Comput.* **22** (2012), no. 6, 1250058.

Quantales and inverse quantal frames

A *quantale* is a sup-lattice equipped with a binary multiplication operation such that multiplication distributes over arbitrary suprema:

$$a(\vee_{i\in I} b_i) = \vee_{i\in I}(ab_i), \ (\vee_{i\in I} b_i)a = \vee_{i\in I}(b_i a)$$

Let Q be a unital involutive quantale with the multiplicative unit e. An element $a \in Q$ is called a partial unit if $aa^*, a^*a \leq e$.

A inverse quantal frame (notion introduced by P. Resende) is a unital involutive quantale Q, such that:

- $a1 \wedge e = aa^* \wedge e$ for all $a \in Q$,
- $a = (a1 \wedge e)a$ for all $a \in Q$,
- every element of Q is a join of partial units.

The element $a1 \wedge e$ is called the support of a.

Example

Let $G = X \times X$ be the pair groupoid, where X is a set. Then the powerset $\mathcal{P}(G)$ is an inverse quantal frame. Indeed, if $a \in \mathcal{P}(G)$ then

$$a1 \wedge e = \{(x,x) \colon \text{ there is } y \text{ such that } (x,y) \in a\}.$$

Partial units of $\mathcal{P}(G)$ are precisely the bisections which are just the elements of the symmetric inverse semigroup $\mathcal{I}(X)$.

We have the following three objects, each of which allows to recover the other two:

1. The groupoid $X \times X$.
2. The inverse quantale frame $\mathcal{P}(X \times X)$.
3. The symmetric inverse semigroup $\mathcal{I}(X)$.

Equivalence of pseudogroups and inverse quantal frames

Let S be a pseudogroups. Then the set $S^{\vee}$ of all compatibly closed downsets of S is an inverse quantal frame. It is called the enveloping inverse quantal frame of S.

Example: $(\mathcal{I}(X))^{\vee} = \mathcal{P}(X \times X)$.

Let Q be an inverse quantal frame. Then the set of partial units $\mathcal{I}(Q)$ forms a pseudogroup, called the pseudogroup of partial units of Q.

Example: $\mathcal{I}(\mathcal{P}(X \times X)) = \mathcal{I}(X)$.

These assignments give rise to an equivalence of categories of pseudogroups and inverse quantal frames (in fact, several equivalences with various morphisms, specified below).

Morphisms

Morphisms of pseudogroups: (We consider only those morphisms of pseudogroups whose restriction to idempotents is a frame morphism!)

1. semigroup homomorphisms:
   a) all allowed
   b) proper: inverse image of an ultrafilter is not empty in spatial case
2. meet-preserving semigroup homomorphisms:
   a) all allowed
   b) proper

Morphisms of inverse quantal frames:

1. quantale morphisms: multiplication preserved, suprema preserved:
   a) all
   b) proper: top element preserved
2. quantale morphisms that preserve finite non-empty meets in addition to the above.
   a) all
   b) proper, that is frame morphisms: top (i.e. the empty meet) is preserved.

Internal groupoids: definition

Let C be a category with pullbacks. An internal groupoid is a pair (G_1, G_0), where G_1 is the object of arrows and G_0 is the object of units, equipped with the following structure morphisms:

- $d, r : G_1 \to G_0$ — the *domain* and the range morphisms, respectively.
- $m : G_1 \times_{G_0} G_1 \to G_1$ — the multiplication morphism, where $G_1 \times_{G_0} G_1$ is the pullback of d and r, called the object of composable pairs,
- $u : G_0 \to G_1$, the unit morphism,
- $i : G_1 \to G_1$, the inversion morphism.

These morphisms are subject to some conditions which express axioms of a groupoid.

Localic and topological groupoids

A localic groupoid is an internal groupoid in the category of locales.
A topological groupoid is an internal groupoid in the category of topological spaces. The space of composable pairs is the space

$$G \times_{G_0} G = \{(a, b) \in G \times G : r(a) = d(b)\}.$$

If (a, b) is a composable pair we write $a \cdot b$ or just ab for $m(b, a)$. The axioms are:

1. $du = ru = 1$ — specification of the domain and the map of identity arrows.
2. Let $a \in G$. Then $(ud(a), a)$ is a composable pair and $a \cdot ud(a) = a$. Similarly, $(a, ur(a))$ is a composable pair and $ur(a) \cdot a = a$.
3. If (a, b) is a composable pair, then $d(ab) = d(b)$ and $r(ab) = r(a)$.
4. If a, b, c are such that (a, b) and (b, c) are composable pairs then (ab, c) and (a, bc) are composable pairs and $ab \cdot c = a \cdot bc$ — associativity of the multiplication.
5. $di = r, ri = d$ — inversion switches domain and ranges of arrows.
6. $aa^{-1} = ur(a)$ and $a^{-1}a = ud(a)$ — product of an arrow with its inverse equalsthe identity arrow on the range (or on the domain, if multiplied in the opposite direction).

Inverse quantale frames, localic étale groupoids and pseudogroups

- Q — inverse quantal frame $\Rightarrow \mathcal{G}(Q)$ — localic groupoid of opens.
- G — localic étale groupoid $\Rightarrow \mathcal{O}(G)$ — inverse quantal frame.

Theorem, Resende 2007

1. Let Q be an inverse quantal frame. Then $Q \simeq \mathcal{O}(\mathcal{G}(Q))$.
2. Let G be a localic étale groupoid. Then $G \simeq \mathcal{G}(\mathcal{O}(G))$.

If Q is spatial then then $\mathcal{G}(Q)$ can be replaced by the topological groupoid of completely prime filters of Q and the constructions can be carried out topologically. Applying the topological approach, M. V. Lawson and D. H. Lenz connected spatial pseudogroups with sober étale groupoids (objects and meet-preserving morphisms).

Morphisms of internal groupoids

Let G and G' be internal groupiods in a category C. A morphism $f : G \to G'$ is a pair (f, f_0), where

- $f : G \to G'$ and $f_0 : G_0 \to G_0'$ are morphisms of C
- f and f_0 commute in a natural way with the structure morphisms of the groupoids.

As a consequence, f_0 is uniquely determined by f: $f_0 = d' \cdot f_1 \cdot u = r' \cdot f_1 \cdot u$.

Morphisms of topological groupoids

Let $G = (G, G_0)$ and $G' = (G', G_0')$ be topological groupoids. By a *morphism* $f : G \to H$ we will mean a pair $f = (f, f_0)$, where $f : G \to H$ and $f_0 : G_0 \to H_0$ are continuous maps. Moreover, these maps commute with the structure maps so that the following axioms hold:

1. $f_0 d = d' f$, $f_0 r = r' f$.
2. $fi = i' f$.
3. $fu = u' f$.
4. $fm = m'(f \times f)$.

Morphisms: example

Let G be a group which we consider as a discrete groupoid. Let $f : G \to G$ be a group homomorphism. We have $\mathcal{O}(G) = \mathcal{P}(G)$.

Observation

f^{-1} induces an inverse quantale frame morphism if and only if f is an isomorphism.

f must be one-to-one since $f^{-1}(e) = e$ under a quantale morphism (preservation of multiplicative unit).

Let $a \in G$ is such that $a \notin \mathrm{im}(f)$. Then $a^{-1} \notin \mathrm{im}(f)$ as well, so that $f^{-1}(a) = f^{-1}(a^{-1}) = \varnothing$. But $f^{-1}(aa^{-1}) = f^{-1}(e) = e$. Hence f is onto.

Note also that if $f : G_1 \to G_2$ is a morphism of groupoids then $f^{-1} : \mathcal{O}(G_2) \to \mathcal{O}(G_1)$ preserves finite meets.

So to obtain quantale morphisms that do not preserve meets one should allow laxer morphisms at the topological side.

A characterization of inverse quantal frame morphisms

Let G, G' be localic étale groupoids. A relational morphism $g = (g, g_o)$ from G to G' is a pair $g^* = (g^*, g_0^*)$ where $g^* : \mathcal{O}(G') \to \mathcal{O}(G)$ is a sup-lattice morphism and $g*_0 : \mathcal{O}(G_0') \to \mathcal{O}(G_0)$ is a frame morphism, such that g is agreed with the structure maps of G and G'.

Observation (P. Resende)

For any $a, b \in \mathcal{O}(G')$ we have $g^*(a)g^*(b) \leq g^*(ab)$.

Definition

We say that

- g^* preserves partial units if $g^*(a)$ is a partial unit whenever a is.
- g^* respects supports if $g^*(\zeta'(a)) = \zeta(g^*(a))$ for all partial units $a \in G'$, where ζ' is the support of G' and ζ is the support of G.

A characterization of quantale frame morphisms

Proposition

Let G, G' be localic étale groupoids and let $g = (g, g_0) : G \to G'$ be a relational morphism. The following statements are equivalent:

1. $g : G \to G'$ *preserves partial units and respect the support maps.*
2. $g^* : \mathcal{O}(G') \to \mathcal{O}(G)$ *preserves quantale multiplication.*
3. *The restriction of* g^* *to the partial units of* $\mathcal{O}(G')$ *is a well-defined pseudogroup morphism from* $\mathcal{I}(\mathcal{O}(G'))$ *to* $\mathcal{I}(\mathcal{O}(G))$.

Relational morphisms of topological groupoids

Let G, H be topological groupoids and let $f : G \to \mathcal{P}(H)$ be a map. We call it multivalued map from G to H. For $A \subseteq H$ we define its *inverse image* $f^{-1}(A) = \{x \in G \colon f(x) \cap A \neq \varnothing\}$. We call f continuous if inverse images of open sets are open sets.

Definition

A relational morphism $f : G \to H$ is a pair $f = (f, f_0)$, where $f : G \to H$ is a continuous multivalued map and $f_0 : G_0 \to H_0$ is a continuous map. These maps are required to commute with the structure maps.

- $f : G \to H$ is called star-injective, provided that $a \neq b$ and $d(a) = d(b)$ imply $f(a) \cap f(b) = \varnothing$.
- $f : G \to H$ is called star-surjective, provided that if $e \in H_0$ is such that $e = f_0(f)$ for some $f \in G_0$ and if $a \in H$ is such that $d(a) = e$ then there is some $b \in G$ such that $a \in f(b)$.
- A star-injective and star-surjective relational morphism will be called a relational covering morphism.

Correspondence for morphisms

Theorem, GK and M.V. Lawson

Let G, G' be sober topological étale groupoids and $g = (g, g_0) : G \to G'$ be a relational morphism. The following statements are equivalent:

1. g is a relational covering morphism.
2. $g^* = g^{-1} : O(G') \to O(G)$ is a morphism of inverse quantal frames.
3. The restriction of g^* to $\mathcal{I}(O(G'))$ is a morphism of pseudogroups.

Correspondence for morphisms

Corollary

Let G, G' be as above. The following statements are equivalent:

1. g is a relational covering morphism and $|g(a)| \leq 1$ for all $a \in G$.
2. $g^* = g^{-1} : O(G') \to O(G)$ is a morphism of inverse quantal frames that preserves non-empty finite meets.
3. The restriction of g^* to $\mathcal{I}(O(G'))$ is a meet-preserving morphism of pseudogroups.

Correspondence for morphisms

Corollary (Lawson and Lenz)

Let G, G' be as above. The following statements are equivalent:

1. g is a (non-relational) covering morphism.
2. $g^* = g^{-1} : O(G') \to O(G)$ is a morphism of inverse quantal frames that preserves all finite meets.
3. The restriction of g^* to $\mathcal{I}(O(G'))$ is a meet-preserving proper morphism of pseudogroups.

Relational covering morphisms of localic groupoids

Let G, G' be localic groupoids. A relational covering morphism $f : G \to G'$ will be called a relational covering morphism if f^* preserves partial units and respects supports.

Resende's theorem with morphisms added

1. The category of inverse quantal frames is dually equivalent to the category of localic étale groupoids and their relational covering morphisms.
2. The category of inverse quantal frames and their morphisms that preserve finite meets is equivalent to the category of localic étale groupoids and their (non-relational) covering morphisms.