Non-commutative Stone duality: from analysis to algebra

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1. Stone, 1937

In 1937, Marshall Stone wrote a paper

M. H. Stone, Applications of the theory of Boolean rings to general topology, *Transactions of the American Mathematical Society* **41** (1937), 375–481.

in which he generalized the theory of *finite* Boolean algebras to *arbitrary* Boolean algebras.

This theory is now known as Stone duality.

Stone was what we would now term a functional analyst.

Question: How did he become interested in Boolean algebras?

Answer: Through algebras of commuting projections.

More generally . . .

Let R be a commutative ring.

Denote by $\mathsf{E}(R)$ the set of idempotents of R.

On the set $\mathsf{E}(R)$ define

$$a \wedge b = a \cdot b \quad a \vee b = a + b - ab \quad a' = 1 - a.$$

Theorem $(\mathsf{E}(R), \wedge, \vee, {}', 0, 1)$ *is a Boolean algebra and every Boolean algebra arises in this way.*

2. Stone duality

Stone's work is the first deep result on Boolean algebras.

It is also represents the first construction of a topological space from *algebraic data.*

Define a *Boolean space* to be a 0-dimensional, compact Hausdorff space.

Theorem [Stone, 1937]

1. *Let S be a Boolean space. Then the set* $\mathsf{B}(S)$ *of clopen subsets of S is a Boolean algebra.*

2. *Let A be a Boolean algebra. Then the set* $\mathsf{X}(A)$ *of all ultrafilters of A can be topologized in such a way that it becomes a Boolean space. It is called the* Stone space *of A.*

3. *If S is a Boolean space then* $S \cong \mathsf{XB}(S)$.

4. *If A is a Boolean algebra then* $A \cong \mathsf{BX}(A)$.

Examples

1. Up to isomorphism, there is exactly one countable, atomless Boolean algebra. It is innominate so I call it the *Tarski algebra*. The Stone space of the Tarski algebra is the *Cantor space.*

2. The Stone space of the powerset Boolean algebra P(X) is the Stone-Čech compactification of the discrete space X.

3. Renault, 1980

Renault's monograph

J. Renault, *A groupoid approach to C^*-algebras*, LNM 793, Springer-Verlag, 1980.

highlighted the important role played by inverse semigroups in the theory of C^*-algebras.

Recall that . . .

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

An inverse semigroup S is equipped with two important relations:

1. $s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

2. $s \sim t$ if and only if st^{-1} and $s^{-1}t$ both idempotents. Called the *compatibility relation*. It controls when pairs of elements are *eligible* to have a join.

Example Symmetric inverse monoids $I(X)$ are the prototypes of inverse semigroups just as the symmetric groups are the prototypes of groups.

If X has n elements we sometimes denote the symmetric inverse monoid on n letters by I_n.

The idempotents of an inverse semigroup form a commutative subsemigroup but are not (ring theorists beware!) central.

There were earlier papers on the interactions between inverse semigroups and functional analysis:

B. A. Barnes, Representations of the l_1-algebra of an inverse semigroup, *Trans. Amer. Math. Soc.* **218** (1976), 361–396.

But since Renault's book, inverse semigroups have become a feature of the theory of C^*-algebras.

The work of Ruy Exel is particularly noteworthy

`http://mtm.ufsc.br/~exel/`.

Question: Why inverse semigroups and C^*-algebras?

Answer: Because the set of partial isometries of a C^*-algebra is *almost* an inverse semigroup.

The following is Theorem 4.2.3 of my book on inverse semigroups.

Theorem *The set of partial isometries of a C^*-algebra forms an ordered groupoid*

4. Boolean inverse monoids

Inverse semigroups might not, however, be the right structures to study in this context.

A *Boolean inverse monoid* is an inverse monoid satisfying the following conditions:

1. The set of idempotents forms a Boolean algebra under the natural partial order.

2. Compatible pairs of elements have a join.

3. Multiplication distributes over the compatible joins in (2).

Symmetric inverse monoids are Boolean.

The compatible joins give rise to a (partially) *additive structure.*

Theorem [Wehrung, 2017] *Let S be an inverse submonoid of the multiplicative monoid of a C^*-algebra R where $s^{-1} = s^*$ for each $s \in S$. Then there is a Boolean inverse monoid B such that $S \subseteq B \subseteq R$.*

Example Let S be the monoid that consists of the matrix units in $R = M_n(\mathbb{C})$ together with the zero and the identity. Then B is the Boolean inverse monoid of *rook matrices* in R. The monoid B is isomorphic to the symmetric inverse monoid on n letters.

We view Boolean inverse monoids as non-commutative generalizations of Boolean algebras.

Boolean inverse monoids are 'ring-like' with the partial join operation being analogous to the addition in a ring. Wehrung (2017) proved they form a variety and have a Mal'cev term.

This raises the question of generalizing Stone duality to a non-commutative setting.

What, then, are the generalizations of Boolean spaces?

5. Etale groupoids

We shall regard groupoids as algebraic structures with a subset of *identities*. If G is a groupoid, its set of identities if G_o.

Examples

1. Groups are the groupoids with exactly one identity.

2. Equivalence relations can be regarded as principal groupoids; the *pair groupoid* $X \times X$ is a special case.

3. From a group action $G \times X \to X$ we get the *transformation groupoid* $G \ltimes X$.

A *topological groupoid* is a groupoid G equipped with a topological structure in which both multiplication and inversion are continuous.

A topological groupoid is said to be *étale* if the domain map is a local homeomorphism.

WHY ETALE?

If X is a topological space, denote by $\Omega(X)$ the lattice of all open sets of X.

Theorem [Resende, 2006] *Let G be a topological groupoid. Then G is étale if and only if $\Omega(G)$ is a monoid.*

- Etale groupoids are topological groupoids with an algebraic alter ego.
- Etale groupoids should be viewed as generalized spaces (Kumjian, Crainic and Moerdijk . . . .)

6. Non-commutative Stone duality

A *Boolean groupoid* is an étale groupoid whose space of identities is a Boolean space.

Let G be a groupoid. A *partial bisection* is a subset $A \subseteq G$ such that $A^{-1}A, AA^{-1} \subseteq G_o$.

Let G be a Boolean groupoid. The set of compact-open partial bisections of G is denoted by $\mathsf{B}(G)$.

Let S be a Boolean inverse monoid. The set of ultrafilters of S is denoted by $\mathsf{G}(S)$.

Theorem [Lawson & Lenz, Resende]

1. *Let G be a Boolean groupoid. Then* B(G) *is a Boolean inverse monoid.*

2. *Let S be a Boolean inverse monoid. Then* G(S) *is a Boolean groupoid, called the* Stone groupoid *of S.*

3. *If G is a Boolean groupoid then $G \cong$* GB(G)*.*

4. *If S is a Boolean inverse monoid then $S \cong$* BG(S)*.*

Example

An inverse semigroup is *fundamental* if the only elements centralizing the idempotents are idempotents. A Boolean inverse monoid is *simple* if it has no non-trivial *additive* ideals.

Theorem

1. The finite, fundamental Boolean inverse monoids are finite direct products

   $$I_{n_1} \times \ldots \times I_{n_r}.$$

   [Compare finite dimensional C^*-algebras.]

2. The finite simple Boolean inverse monoids are the finite symmetric inverse monoids $I(X)$.

3. The Boolean groupoid associated with $I(X)$ is the pair groupoid $X \times X$.

7. Applications

- The groups of units of Boolean inverse monoids are the topological full groups. These form an interesting class of infinite groups generalizing the finite symmetric groups.

- Boolean inverse monoids can be used to co-ordinatize MV algebras.

- There are families of Boolean inverse monoids that parallel families of C^*-algebras: AF inverse monoids, Cuntz inverse monoids, ... with the associated groupoids being the groupoids used to construct the C^*-algebras in question.

- Boolean inverse monoids used by Donsig, Fuller and Pitts to obtain a new proof of classical results by Feldman and Moore on von Neumann algebras. Key role played by the cohomology of Boolean inverse monoids (arXiv:1409.1624v2).

8. Envoi

1. Develop the theory of Boolean inverse monoids as the non-commutative theory of Boolean algebras. *For example*, the Booleanization of an inverse semigroup has Paterson's universal groupoid as its Stone groupoid.

2. Develop the theory of Boolean inverse monoids by analogy with (is there more going on here?) the theory of C^*-algebras of real rank zero. Observe that the analogue of the Cuntz C^*-algebra $\mathcal{O}_2$ is the Cuntz inverse monoid C_2. The group of units of C_2 is Thompson's group V.

3. Classify Boolean inverse monoids using the homology theory of their associated Stone groupoids.

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