Non-commutative Stone duality: étale categories and their monoids

This is joint work with Ganna Kudryavtseva and appeared in Adv. Math 311 (2017), 378–468.

0. Motivations for this talk.
1. Some background on pointless spaces.
2. Etale topological categories.
3. Restriction quantal frames.
4. Complete restriction monoids.

What next?

0. Motivations for this talk

- The paper by T. Bice,
  Noncommutative Pierce duality between Steinberg rings and ample ringoid bundles.

  arXiv: 2012.03006v3.

This paper uses what GK and MVL did in their paper for Adv. Math.

- What might we mean by a non-commutative topological space?

The answer to the question above might be

étale topological groupoid

but this contains an involution — important for constructing a C^*-algebra but not natural for an honest-to-goodness topological space even if it is non-commutative.

$\Longrightarrow$ Work with étale topological categories. (whatever they are)

1. Some background on pointless spaces

We usually take the points as primary and the open sets as secondary (certain sets of points).

We reverse the order and take the open sets as primary and the points as secondary (how this is done and the problems encountered will be described later).

Let X be a topological space. Denote the lattice of open sets of X by $\Omega(X)$.

<u>Definition.</u> A *frame* is a complete lattice in which finite meets distribute over arbitrary joins.

<u>Example</u> $\Omega(X)$ is a frame, but we are <u>NOT</u> saying that all frames arise in this way. Those that do are called *spatial*.

What do we mean by the points of a frame?

<u>Motivation</u> Let X be a topological space and let $x \in X$. Denote by $\mathcal{O}_x$ the set of all open sets of X that contain x.

<u>Definition</u> A <u>point</u> of a frame F is defined to be any <u>completely prime filter</u> in F. The set of such points in F is denoted $\underline{pt}(F)$.

<u>Remark</u> Every point of a topological space X determines a completely prime filter but different points can determine the same filter and not all completely prime filters come from points. Spaces where neither of these events occur are called *sober*.

Let F be a frame. Then $pt(F)$ becomes a topological space when we define X_a = all completely prime filters that contain $a \in F$.

$$\{ X_a : a \in F \}$$

is a topology on F.

The opposite (purely formal) of the category of frames is called the category of locales.

Topological spaces and locales are different ways of describing spaces. They are not equivalent. See P. T. Johnstone, Stone spaces, CUP, 1986.

2. Etale topological categories

A topological category C is said to be *étale* if the functions $\underline{d}$ and $\underline{r}$ are local homeomorphisms.

Why should you care about étale-ness?

Because $\Omega(C)$ is a monoid with identity C_0.

We shall now describe the monoids $\Omega(C)$.

A monoid S is called an Ehresman semigroup if there is a set of commuting idempotents $\underline{P}(S) \subseteq \underline{E}(S)$, called projections, equipped with two functions $S \to \underline{P}(S)$, denoted by $a \mapsto a^*$ and $a \mapsto a^+$ s.t. $aa^* = a$, $a^+a = a$, the operations $*$ and $+$ are the identity on $\underline{P}(S)$ and $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$.

Let S be an Ehresman semigroup.

Define $a \overline{L} b \iff a^* = b^*$ and

$a \overline{R} b \iff a^+ = b^+$.

<u>Claim</u> $a \overline{L} b \implies ac \overline{L} bc$

That is $\overline{L}$ is a <u>right congruence</u>.

<u>Proof</u> $a \overline{L} b$ means $a^* = b^*$.

Thus $(ac)^* = (a^*c)^* = (b^*c)^*$

$= (bc)^*$. $\blacksquare$

A quantale is a lattice-ordered semigroup. A quantal frame is a quantale where $(Q, \leq)$ is a frame.

A partial isometry in an Ehresmann quantal frame is an element a s.t

$$b \leq a \implies b = ab^* = b^+ a$$

A restriction quantal frame is an Ehresmann quantal frame in which the top element is a join of partial isometries and the set of partial isometries is closed under multiplication.

<u>Theorem</u> If C is an étale category then $\Omega(C)$ is a restriction quantal frame.

We shall now go in the opposite direction from restriction quantal frames to étale topological categories.

3. Restriction quantal frames

Let Q be a restriction quantal frame with top element 1 and identity e. Define $\underline{C}(Q)$ to be the set of all completely prime filters of Q.

If A is a completely prime filter define

$$\underline{d}(A) = (A^*)^\uparrow \quad \text{and}$$

$$\underline{r}(A) = (A^+)^\uparrow$$

Define $A \cdot B = (AB)^\uparrow$ if

$\underline{d}(A) = \underline{r}(B)$.

<u>Proposition</u> Let Q be a restriction quantal frame. Then $(\mathsf{C}(Q), \cdot)$ is a category

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- The identities of this category have the form $F^{\uparrow}$ where $F \subseteq e^{\downarrow}$ is a completely prime filter.

- If A is a completely prime filter in Q then

$$A = (aF)^{\uparrow}$$

where a is a partial isometry; $a^* \in F$; $F \subseteq e^{\downarrow}$ is a completely prime filter

Looks like a "coset"

Let Q be a restriction quantal frame. Define

$\mathcal{X}_a$ = all completely prime filters of Q that contain a.

Put $\tau = \{\mathcal{X}_a : a \in PI(Q)\}$

partial isometries of Q

Theorem Let Q be a restriction quantal frame. Then τ is the base for a topology on $\underline{C}(Q)$ with respect to which $\underline{C}(Q)$ is an étale topological category.

Example Let X be a finite non-empty set. Then $X \times X$ is a category (in fact, a groupoid) when we define $d(x,y) = (y,y)$, $r(x,y) = (x,x)$ and

$$(x,y)\cdot(y,z) = (x,z).$$

$P(X \times X)$ is a restriction quantal frame. Its elements are binary relations on X. The partial isometries are the binary relations of partial bijections.

We may define sober étale categories and spatial restriction quantal frames.

Restriction quantal frames are complicated objects. Can we construct them from something simpler? Yes!

4. Complete restriction monoids

Let S be an Ehresmann monoid with identity e where $P(S) = e^{\downarrow}$.

We say that S is a *restriction monoid* if

$$af = (af)^{+}a \quad \text{and} \quad fa = a(fa)^{*}$$

for all elements a and projections f. Let $a, b \in S$, a restriction monoid. We say a and b are *compatible*, $a \sim b$, if

$$ab^{*} = ba^{*} \quad \text{and} \quad b^{+}a = a^{+}b.$$

If S is a restriction monoid then we may define a partial order $\leq$ on S by

$$a \leq b \text{ iff } a = ba^* = a^+b.$$

<u>Definition</u> A complete restriction monoid is a restriction monoid in which every compatible subset has a join and multiplication distributes over such joins.

<u>Example</u> If Q is a restriction quantal frame then $PI(Q)$ is a complete restriction monoid.

In fact, the above example is typical.

<u>Theorem</u> For suitable definitions of morphisms, the category of complete restriction monoids is equivalent to the category of restriction quantal frames.

<u>Example</u> Let S be a pseudogroup. Define

$$a^* = a^{-1}a \quad \text{and} \quad a^+ = a a^{-1}$$

Then with respect to these definitions, S is a complete restriction monoid. The étale category $\underline{C}(S)$ is, in fact, an étale groupoid. In this way, we recover the usual "duality" between pseudogroups and étale groupoids.

What next?

Etale topological categories are exemplars of "non-commutative" topological spaces. Can they be used to construct a theory of "non-commutative sheaves"? How does this relate to Bice's work?