Boolean full groups

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In collaboration with

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1. Background

1. T. Giordano, I. F. Putnam, C. F. Skau, Full groups of Cantor minimal systems, *Israel J. Math.* **111** (1999), 285–320. Interesting groups arising from dynamical systems called *topological full groups*.

2. H. Matui in a sequence of important papers generalized (1) to the setting of étale topological groupoids and their associated topological full groups. The classical Thompson groups arise in this way.

3. V. Nekrashevych introduced the term *groups of dynamical origin* and suggested that they may be viewed as infinite generalizations of finite symmetric groups.

My (semigroup-based) interest is as follows:

1. We have developed a non-commutative generalization of classical Stone duality (Boolean algebras $\longleftrightarrow$ Boolean spaces) linking étale groupoids and inverse monoids.

2. The groups of units of such inverse monoids are the topological full groups.

3. The inverse monoids which arise are related to C^*-algebras of real rank zero such as AF, Cuntz and Cuntz-Krieger as well as those arising from aperiodic tilings

The goal of my talk is to explain these connections.

2. Motivating example

CLAIM: *The finite symmetric groups* S_n *arise as the groups of units of the finite symmetric inverse monoids* I_n *and the structure of the latter has an impact on the structure of the former.*

3. Boolean inverse monoids

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Elements of the form $a^{-1}a$ and aa^{-1} are idempotents.

Not obvious but the idempotents in an inverse semigroup commute and so form a *meet-semilattice*. We refer to the *semilattice of idempotents*.

The *symmetric inverse monoid* $I(X)$ of all partial bijections of X really is an inverse monoid.

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation* $\leq$ *is a partial order with respect to which the inverse semigroup is a partially ordered semigroup.*

It is called the *natural partial order*.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a *necessary condition* for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A non-empty subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

In the symmetric inverse monoid $I(X)$ we have the following:

- The idempotents in $I(X)$ are the identity functions defined on the subsets of X. Denote them by 1_A, where $A \subseteq X$, called *partial identities*. Then

  $$1_A \leq 1_B \Longleftrightarrow A \subseteq B$$

  and

  $$1_A 1_B = 1_{A \cap B}.$$

  Thus the semilattice of idempotents on $I(X)$ is isomorphic to $\mathcal{P}(X)$.

- The natural partial order is the restriction order. Partial bijections f and g are compatible if and only if $f \cup g$ is a partial bijection.

Theorem [Wagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each non-empty finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse semigroup is said to be *Boolean* if it is distributive and its semilattice of idempotents is a (generalized) Boolean algebra. (In this talk, I shall only deal with unital Boolean algebras).

- An inverse semigroup that has all binary meets is called a *meet-semigroup*.

Idea

Algebra	Topology
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

Commutative	Non-commutative
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet-semigroup

In this talk, I will concentrate on *Boolean inverse monoids*. Symmetric inverse monoids are Boolean.

3. Non-commutative Stone duality

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'.

Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms an inverse monoid.*

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

Etale groupoids therefore have a strong algebraic character.

A *compact Boolean space* is a compact Hausdorff space with a basis of clopen subsets.

Theorem [Classical Stone duality] *The category of Boolean algebras is dually equivalent to the category of Boolean spaces.*

Example Under classical Stone duality the Cantor space corresponds to what I shall call the *Tarski algebra* — the unique countable atomless Boolean algebra.

A *Boolean groupoid* is an étale topological groupoid whose space of identities is a compact Boolean space.

If G is a Boolean groupoid denote by $\mathsf{KB}(G)$ the set of all compact-open local bisections.

If S is a Boolean inverse monoid denote by $\mathsf{G}(S)$ the set of ultrafilters of S.

Theorem [Non-commutative Stone duality]

1. $\mathsf{KB}(G)$ *is a Boolean inverse monoid.*

2. $\mathsf{G}(S)$ *is a Boolean groupoid.*

3. *Boolean inverse monoids are in duality with Boolean groupoids.*

4. *Boolean inverse meet-monoids are in duality with Hausdorff Boolean groupoids.*

INTERMISSION

An inverse semigroup is *fundamental* if the only elements that centralize all idempotents are themselves idempotents. Example: symmetric inverse monoids are fundamental.

Theorem [Wagner] *An inverse semigroup is fundamental if and only if it is isomorphic to an inverse semigroup of partial homeomorphisms between the open subsets of a T_0 space where the domains of definition of the elements form a basis for the space.*

Fundamental inverse semigroups should therefore be viewed as inverse semigroups of partial homeomorphisms.

A *closed ideal* in a Boolean inverse monoid is an (semigroup) ideal closed under finite compatible joins. A Boolean inverse monoid is 0-*simplifying* if it contains no non-trivial closed ideals.

Example Finite symmetric inverse monoids are fundamental and 0-simplifying.

A topological groupoid is said to be *effective* if the interior of the isotropy subgroupoid is just the space of identities. Such a groupoid is *minimal* if there are no non-trivial open invariant subsets.

Theorem [More non-commutative Stone duality] *Under non-commutative Stone duality, we have that*

1. *Fundamental Boolean inverse monoids correspond to effective étale groupoids.*

2. 0*-simplifying Boolean inverse monoids correspond to minimal étale groupoids.*

GROUPS!

4. Groups, inverse semigroups and groupoids

Definition A Boolean inverse monoid that is both fundamental and 0-simplifying is said to be *simple*.

Theorem [The simple alternative] *A simple Boolean inverse monoid is either isomorphic to a finite symmetric inverse monoid or atomless.*

Corollary *A simple countable Boolean inverse monoid has the Tarski algebra as its set of idempotents.*

Definition Denote by Homeo($\mathscr{S}$) the group of homeomorphisms of the Boolean space $\mathscr{S}$. By a *Boolean full group*, we mean a subgroup G of Homeo($\mathscr{S}$) satisfying the following condition: let $\{e_1, \dots, e_n\}$ be a finite partition of $\mathscr{S}$ by clopen sets and let $g_1, \dots, g_n$ be a finite subset of G such that $\{g_1 e_1, \dots, g_n e_n\}$ is a partition of $\mathscr{S}$ also by clopen sets. Then the union of the partial bijections $(g_1 \mid e_1), \dots, (g_n \mid e_n)$ is an element of G. We call this property *fullness* and term *full* those subgroups of Homeo($\mathscr{S}$) that satisfy this property.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups.*

2. *Simple Boolean inverse monoids*

3. *Minimal, effective Boolean groupoids.*

Let $\mathscr{S}$ be a compact Hausdorff space. If $\alpha \in \text{Homeo}(\mathscr{S})$, define

$$\text{supp}(\alpha) = \text{cl}\{x \in \mathscr{S} \colon \alpha(x) \neq x\}$$

the *support* of α.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups in which each element has clopen support.*

2. *Simple Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff Boolean groupoids.*

A groupoid is *principal* if it is defined from an equivalence relation.

A Boolean inverse monoid is *basic* if each non-zero element is a finite join of some infinitesimals and possibly an idempotent.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups in which each element has a* clopen fixed-point set*.*

2. *Simple basic Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff, principal Boolean groupoids.*

Example There is a family $C_2, C_3, \ldots$ of simple, countable atomless Boolean inverse monoids, the *Cuntz inverse monoids*, whose groups of units are the Thompson groups $V_2, V_3, \ldots$, respectively.

The groupoid associated with C_n is the same as the groupoid associated with the Cuntz C^*-algebra $\mathcal{O}_n$.

Representations of the inverse monoids C_n are (unknowingly) the subject of *Iterated function systems and permutation representations of the Cuntz algebra* by O. Bratteli and P. E. T. Jorgensen, AMS, 1999.

Example The *AF monoids* are a class of fundamental Boolean inverse monoids defined to be direct limits $\varinjlim S_i$ where the inverse semigroups S_i are finite direct products of finite symmetric inverse monoids and the maps between them preserve joins.

Their groups of units are direct limits of finite direct products of finite symmetric groups with morphisms being by means of diagonal embeddings.