Higher dimensional generalizations of the Thompson groups via higher rank graphs

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In collaboration with Alina Vdovina and Aidan Sims. This talk is based on the arXiv paper with the same name: arXiv:2010.08960.

1. Idea

We want to construct groups.

Groups are abstract versions of groups of bijections.

How should we construct bijections?

Construct bijections by gluing together *partial bijections*.

But, partial bijections can only be glued together if they are *compatible*.

Thus, we can construct bijections by gluing together compatible sets of partial bijections.

Where do partial bijections come from?

One possible source of examples is provided by cancellative monoids. For example, we can multiply by an element on the left — this is a partial bijection.

More generally, we can replace cancellative monoids by cancellative categories (where we view a category as a 'monoid with many identities').

The goal of this talk is therefore to show how to construct groups from certain cancellative categories.

The cancellative categories in question are the *higher rank graphs*.

These really are categories (not graphs).

They generalize free categories, in that each element of the category is assigned a ‘length’ (better: ‘size’) not from $\mathbb{N}$, as in free categories, but from $\mathbb{N}^k$ instead.

2. Inverse semigroups as the abstract theory of partial bijections.

As groups are to bijections, so inverse semigroups are to partial bijections.

> "Symmetry denotes that sort of concordance of several parts by which they integrate into a whole." – Hermann Weyl

Inverse semigroups arose by abstracting *pseudogroups of transformations* in the same way that groups arose by abstracting groups of transformations. There were three independent approaches: Gordon B. Preston (1925–2015) in the UK; Charles Ehresmann (1905–1979) in France; Viktor V. Wagner (1908–1981) in the USSR.

They all three converge on the definition of 'inverse semigroup'.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Observe that aa^{-1} and $a^{-1}a$ are idempotents.

The idempotents in an inverse semigroup always commute with each other.

Groups are the inverse semigroups having exactly one idempotent.

The image of an inverse semigroup under a semigroup homomorphism is always inverse. If θ is a semigroup homomorphism with domain an inverse semigroup and $\theta(a)$ is an idempotent, then there is an idempotent e such that $\theta(e) = \theta(a)$.

Define $a \leq b$ if $a = ba^{-1}a$. This is a partial order called the *natural partial order*. It has some nice properties:

- If $a \leq b$ and $c \leq d$ then $ac \leq bd$.

- If $a \leq b$ then $a^{-1} \leq b^{-1}$.

- If $a \leq e$ and e is an idempotent then a is an idempotent.

- If $a, b \leq c$ then ab^{-1} and $a^{-1}b$ are idempotents.

Define $a \sim b$, and say that a and b are *compatible*, if $a^{-1}b$ and ab^{-1} are both idempotents.

Example: the symmetric inverse monoid

Let X be a set equipped with the discrete topology. Denote by $\mathcal{I}(X)$ the set of all partial bijections of X. This is an example of an inverse semigroup called the *symmetric inverse monoid*.

- The inverse of the partial bijection f is f^{-1}.

- The idempotents are the identity functions on the subsets of X.

- The product of two idempotents is the idempotent defined on the intersection of their domains of definition.

Crucially, we have the following:

- $f \leq g$ if and only if $f \subseteq g$.
- $f \sim g$ if and only if $f \cup g$ is a partial bijection.

The fact that inverse semigroups really are the abstract theory of partial bijections is expressed by the following which is the analogue of Cayley's theorem.

Theorem [Wagner-Preston] *Every inverse semigroup can be embedded in a symmetric inverse monoid.*

3. Groups from inverse semigroups.

Recall that groups are inverse semigroups with exactly one idempotent.

If S is an inverse semigroup, define $a \, \sigma \, b$ if there exists $c \leq a, b$. If e and f are idempotents then $ef \leq e, f$. So, all idempotents are identified by σ.

Then σ is a congruence on S, the inverse semigroup S/σ is a group, and if ρ is any congruence on S such that S/ρ is a group, then $\sigma \subseteq \rho$.

Thus, σ is the most efficient way of getting a group out of an inverse semigroup.

BUT the problem with the above construction is that if S (always a monoid) contains a zero then the group above is trivial.

This suggests that we look at 'large' elements of S, which exclude the zero. In this talk large means the following.

We say that a non-zero idempotent e is *essential* if $ef \neq 0$ whenever f is a non-zero idempotent. We say that the element $a \in S$ is *essential* if both $a^{-1}a$ and aa^{-1} are essential idempotents.

The *essential part*, S^e, of the inverse semigroup S consists of all the essential elements of S. It is easy to show that S^e is always an inverse subsemigroup of S. Define the group associated with S as follows:

$$\boxed{\mathscr{G}(S) = S^e/\sigma.}$$

4. Inverse semigroups from cancellative categories.

We shall now find a source of examples of inverse semigroups to which we can apply the above construction.

Let C be a category.

Recall that we regard categories as being algebraic structures; thus, every element is an arrow (amongst which are the special arrows called *identities*). The set of identities of C is denoted by C_o.

If $a \in C$ then $\mathbf{d}(a)$ is the unique identity such that $a\mathbf{d}(a) = a$, and similarly $\mathbf{r}(a)a = a$.

If $a, b \in C$ then ab is defined if and only if $\mathbf{d}(a) = \mathbf{r}(b)$.

Think of arrows *comme ça*: $\mathbf{d}(a) \xrightarrow{a} \mathbf{r}(a)$.

A category C is said to be *cancellative* if

$$ab = ac \Rightarrow b = c$$

and

$$ba = ca \Rightarrow b = c.$$

An *invertible element* in a category is an element x for which there exists an element y such that $yx = \mathbf{d}(x)$ and $xy = \mathbf{r}(x)$. A category in which the only invertible elements are the identities is said to be *conical*.

It is convenient to assume that from now on all our categories are cancellative and conical.

A subset $R \subseteq C$ is said to be a *right ideal* if $r \in R$ and $c \in C$ and the product rc is defined implies that $rc \in R$.

If $X \subseteq C$ is any subset then XC is the right ideal *generated* by X. If X is a finite set we say that XC is a *finitely generated* right ideal.

We call aC the *principal right ideal* generated by a.

We say that the category X is *finitely aligned* if $aC \cap bC$ is always a finitely generated right ideal.

Example

Let C be a category. When is it finitely generated as a right ideal?

Suppose, first, that the set of identities, C_o, is finite. Then $C = C_oC$ (since each arrow ends in an identity). Thus C is finitely generated as a right ideal.

Suppose that C is finitely generated as a right ideal. Then there is a finite set X such that $C = XC$. But, then, each identity of C must equal an identity of X, but X is a finite set.

We have proved that a category C is finitely generated as a right ideal if and only if it has a finite number of identities.

Let R_1 and R_2 be right ideals of the category C. A function $\theta\colon R_1 \to R_2$ is a *morphism* if $\mathbf{d}(\theta(r)) = \mathbf{d}(r)$ for all $r \in R_1$ and $\theta(rc) = \theta(r)c$ for all $c \in C$ where the product is defined.

Morphisms are analogous to the homomorphisms between right R-modules in ring theory.

Theorem *Let C be a cancellative finitely aligned category with a finite number of identities. Then* $\mathsf{R}(C)$, *the set of all bijective morphisms between the finitely generated right ideals of C, is an inverse monoid. The group associated with C is*

$$\boxed{\mathscr{G}(C) = \mathsf{R}(C)^e/\sigma.}$$

We can say more about the structure of the inverse monoid $\mathsf{R}(C)^e$.

An inverse semigroup S is said to be *E-unitary* if $e \leq a$, where e is an idempotent, implies that a is an idempotent.

Lemma *An inverse semigroup S is E-unitary if and only if $\sigma = \sim$.*

Proposition *The inverse monoid $\mathsf{R}(C)^e$ is E-unitary.*

5. Projective right ideals.

To say something about the group we get, we need to add more conditions.

Let C be a category. Elements a and b are said to be *independent* if $aC \cap bC = \varnothing$. Otherwise they are said to be *dependent*.

A finite set of independent elements will be called a *code*.

A code X is said to be *maximal* if every element of C is dependent on an element of X

A right ideal generated by a code is said to be *projective*. The idea to study projective right ideals is due to Fountain.

A category C is said to be *strongly finitely aligned* if $aC \cap bC \neq \varnothing$ is projective.

Example

Let A be a finite (non-empty) set that we call an *alphabet*.

The free monoid, A^*, on A consists of all finite strings over the alphabet A with concatenation as the semigroup operation.

It is a category with exactly one identity (the empty string) having a trivial group of units.

If a and b are strings, then either $aA^* \cap bA^* = \varnothing$ or one of a or b is a prefix of the other.

It follows that A^* is finitely aligned. In fact, $aA^* \cap bA^*$ is either empty or of the form cA^*.

By a code is meant a finite prefix code.

Each finitely generated right ideal of A^* is generated by a finite prefix code.

The maximal codes are the maximal finite prefix codes.

Theorem *Let C be a cancellative strongly finitely aligned category with a finite number of identities. Then* P(C)*, the set of all bijective morphisms between the projective right ideals of C, is an inverse monoid. The inverse monoid* P$(C)^e$ *consists of all the morphisms between the projective right ideals generated by maximial codes.*

We now add in one extra assumption.

Theorem *Let C be a cancellative strongly finitely aligned conical category with a finite number of identities.* Suppose that every essential finitely generated right ideal contains an essential projective right ideal. *Then*

$$\boxed{\mathscr{G}(C) = \mathsf{P}(C)^e/\sigma.}$$

6. Higher rank graphs.

We shall now define some categories that satisfy our conditions (and so can be used to build groups).

A countable *category* C is said to be a *higher rank graph* or a k-*graph* if there is a functor $d\colon C \to \mathbb{N}^k$, called the *degree map*, satisfying the *unique factorization property* (UFP): if $d(a) = \mathbf{m} + \mathbf{n}$ then there are unique elements a_1 and a_2 in C such that $a = a_1 a_2$ where $d(a_1) = \mathbf{m}$ and $d(a_2) = \mathbf{n}$. We call $d(x)$ the *degree* of x.

You should regard higher rank graphs as generalizations of free categories: in fact, the 1-graphs are precisely the countable free categories.

It can be proved that, if C is a k-graph, then:

1. C is cancellative.

2. C is conical.

3. The elements of C of degree $\mathbf{0}$ are precisely the identities.

I will add a couple of further assumptions (familiar to C*-algebra theorists).

A higher rank graph C has *no sources* if for each identity e of C and element $\mathbf{m} \in \mathbb{N}^k$ there exists an arrow $x \in C$ such that $\mathbf{r}(x) = e$ and $d(x) = \mathbf{m}$.

A higher rank graph C is *row finite* if for each identity e of C, the number of elements of eC of degree $\mathbf{m}$ is finite.

Example

Let G be a finite directed graph.

Denote by G^* the free category generated by G.

(In the case of the free monoid, G is just a 'bouquet of circles').

Then, G^* has no sources means that the in-degree of every vertex of G is at least 1.

G^* is automatically row finite.

We now have the following results:

- A higher rank graph that is row finite is finitely aligned.

- If $\mathbf{m} \in \mathbb{N}^k$ then the set of all elements of C of degree $\mathbf{m}$ is a maximal code.

- Every finitely generated essential right ideal contains a right ideal generated by a maximal code.

We now come to the main theorem of this talk.

Theorem *Let C be a higher rank graph such that the following properties hold:*

- *C has a finite number of identities.*

- *C has no sources.*

- *C is row-finite.*

Then, we may construct a group $\mathscr{G}(C)$ as $\mathsf{P}(C)^e/\sigma$ which is also isomorphic to $\mathsf{R}(C)^e/\sigma$.

Presumably the group $\mathscr{G}(C)$ is telling us something about the structure of maximal codes in C.

This is where algebra meets geometry.

The following theorem is more advanced and requires a knowledge of étale groupoids.

Theorem *Let C be a higher rank graph as above:*

1. *The group $\mathscr{G}(C)$ is a topological full group.*

2. *If the higher rank graph C is also aperodic and cofinal then $\mathscr{G}(C)$ is a topological full group of an étale groupoid which is Hausdorff, effective and minimal. If $\mathscr{G}(C)$ is countably infinite then it is isomorphic to a subgroup of the group of automorphisms of the Cantor set.*

The second result above justifies (via Matui) the use of K_0-groups of the associated C*-algebras as invariants of the groups.

7. An example.

Let A be an alphabet with exactly two elements. Denote by A^* the free monoid on A (as before).

This is a higher rank graph where the degree functor is simply the length homomorphism.

The group $\mathscr{G}(A^*)$ is the Thompson group V or $G_{2,1}$.

It is the group of units of a simple Boolean inverse $\wedge$-monoid C_2 (called the *Cuntz inverse monoid*). Under *non-commutative Stone duality*, this is isomorphic to the topological full group of a Hausdorff, effective minimal étale groupoid the identity space of which is the Cantor space.

The above example and its connections with classical group theory are discussed in

M. V. Lawson, *The polycyclic inverse monoids and the Thompson groups revisited*, in (P. G. Romeo, A. R. Rajan eds) Semigroups, categories and partial algebras ICSAA 2019, Springer, Proc. in Maths and Stats, volume 345.

The motivation for this example comes from

J.-C. Birget, The groups of Richard Thompson and complexity, *IJAC* **14** (2004), 569–626.

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