What are topological full groups?

Mark V Lawson
HWU

Discovery procedure

Certain subjects predominate in a given area of mathematics but not in others, even though they may be important there.

Example Sheaf theory originated in topology but has gone on to be useful in algebra. For example, the work of Dauns and Hofmann.

In ‘our’ context, topological full groups arose in the study of operator algebras but should be interesting to group theorists.

How should they be defined independently of étale groupoids?

Topological full groups were introduced by Matui. But we shall use the term in the following sense.

A *Boolean space* is a compact Hausdorff space with a basis of clopen sets. A *Boolean groupoid* is an étale topological groupoid whose space of identities is a Boolean space. A *local bisection* of a topological groupoid G is a subset A of G such that $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ (resp. $\mathbf{r}(a) = \mathbf{r}(b)$) implies that $a = b$; this is equivalent to requiring that $A^{-1}A, AA^{-1} \subseteq G_o$, the identities of G. A *bisection* of G is a subset A such that $A^{-1}A = G_o = AA^{-1}$.

A *topological full group* is the group of compact-open bisections of a Boolean groupoid. Using non-commutative Stone duality, these are the groups of units of Boolean inverse monoids.

I came to the theory of étale groupoids mainly through the work of Kellendonk.

However, I also studied the Thompson group V from the vantage point of inverse semigroups.

It is worth noting that Thompson originally constructed his groups starting from λ-calculus.

The categorical way of handling λ-calculi is to use Cartesian closed categories.

Example

I shall define two categories C_1 and C_2 and show how they may be used to construct the Thompson group V.

We shall begin with the category C_2. This has as objects those sets equipped with a binary operation λ and two unary operations α_1 and α_2 satisfying the following two laws:

BCA1 $\lambda(\alpha_1(x), \alpha_2(x)) = x$.

BCA2 $\alpha_i\lambda(x_1, x_2) = x_i$.

Any structure that satisfies these two laws is called a *binary Cantor algebra*.

The category C_2 consists of all binary Cantor algebras and homomorphisms between them.

The category C_2 is an example of a *variety* in the sense of universal algebra.

Varieties always have free objects. Denote the free object on one generator in the variety by $F(x)$.

We have that $V = \mathrm{Aut}(F(x))$.

We now define the category C_1.

The objects of this category are the pairs (A, f) where A is a set and $f\colon A \to A^2$ is a bijection. The homomorphisms are the obvious maps.

It is not obvious, but it can be shown that the category C_1 is Cartesian closed.

This category is known as the *Jónsson–Tarski topos*.

It is not obvous but C_1 and C_2 are isomorphic.

This result was known to Higman in 1974.

As a result of this isomorphism we are dealing with a variety which is also Cartesian closed

Thus we are dealing with a *Cartesian closed variety*.

This variety gives rise to the topological full group V.

This turns out to be typical.

Theorem 1 (Garner/adapted) *All topological full groups arise from Cartesian closed varieties.*

Theorem 2 (Garner/adapted) *If a topological full group occurs in a minimal ample groupoid then in fact the Cartesian closed variety is actually a topos.*

Garner eschews universal algebra and prefers to work with topos theory.

I have many questions, but here is one: what is the analogue of the binary Cantor algebra for the groups that arise from finite directed graphs?

References

1. R. Garner, Cartesian closed varieties I: the classification theorem, arXiv:2302.04402.

2. R. Garner, Cartesian closed varieties II: links to algebra and self-similarity, arXiv:2302.04403.

3. G. Higman, Finitely presented infinite simple groups, *Notes on pure mathematics 8*, (1974), ANU Canberra, Australia.

4. M. V. Lawson, The structure of étale Boolean right restriction monoids, arXiv:2404:08606.

5. M. V. Lawson, *Inverse semigroups: the theory of partial symmetries,* Second Edition, World Scientific, late 2025.