Non-commutative Stone duality

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Classical/commutative Stone duality

A *Boolean space* is a compact, Hausdorff space with a basis of clopen subsets.

Theorem (Stone) *The category of Boolean algebras is dually equivalent to the category of Boolean spaces.*

The above theorem can be generalized by replacing Boolean algebras by *generalized Boolean algebras*.

As an example of the above theorem, the countable atomless Boolean algebra corresponds to the Cantor space.

Non-commutative Stone duality

An étale groupoid is said to be *Boolean* if its space of identities is a Boolean space. (The point is that there are classes of étale groupoids depending on the properties of the space of identities.) By modifying the proof of the classical theorem, we obtain the following.

Theorem (Non-commutative Stone duality) *A category (for some class of morphisms) of Boolean inverse monoids is dually equivalent to a category (for some class of morphisms) of Boolean groupoids.*

The above theorem can be generalized by replacing Boolean inverse *monoids* by Boolean inverse *semigroups*.

The point is that Boolean inverse monoids and Boolean groupoids contain exactly the same information though encoded in different ways.

Whenever you see ‘compact-open G-sets’ or (equivalently) ‘compact-open local bisections’ you are working in the Boolean inverse monoid.

There is a more general result (Lawson and Lenz) which sets up an adjunction between abstract pseudogroups and étale groupoids.

A Boolean inverse monoid is said to be *semisimple* if every element is above (in the natural partial order) a finite number of elements.

An *infinitesimal* in a Boolean inverse monoid is a non-zero element whose square is zero.

A Boolean inverse monoid is *basic* if every non-zero element is a join of infinitesimals.

A Boolean inverse monoid is said to be a *meet monoid* if it has all binary meets.

A Boolean inverse monoid is said to be *fundamental* if the only elements that commute with all idempotents are themselves idempotents.

An ideal in a Boolean inverse monoid is said to be *additive* if it is closed under binary joins.

A Boolean inverse monoid is said to be 0-*simplifying* if the only additive ideals are the trivial ones.

A Boolean inverse monoid is said to be *simple* if it is both 0-simplifying and fundamental.

Boolean inverse monoid	**Boolean groupoid**
Group of units of monoid	Topological full group
Semisimple	Discrete
Meet monoid	Hausdorff
Fundamental	Effective
Basic	Principal and Hausdorff
0-simplifying	Minimal
Simple	Minimal and effective
(Quasi-fundamental)	(Topologically free)

Observe that Matui's topological full group is an image of this one.

As an aside, being 'congruence-free' is related to being 'purely infinite'.

Why bother with Boolean inverse monoids?

- Wehrung (Section 3.2) proved that Boolean inverse monoids form a variety in the sense of universal algebra. This is not obvious since the join is a partial operation.

- Some results are easier to prove for Boolean inverse monoids. For example: the Dichotomy theorem — a 0-simplifying Boolean inverse monoid is either semisimple or atomless.

References

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