Non-commutative Stone duality

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Preview

1. Classical Stone duality.
2. Ideas behind non-commutative Stone duality.
3. Non-commutative Stone duality.
4. The étale case.
5. Other special cases.
6. Applications.
7. References.

1. Classical Stone duality

A *Boolean algebra* is an algebraic structure $(B, +, \cdot, ^{-}, 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following axioms:

(B1) $(x + y) + z = x + (y + z)$.

(B2) $x + y = y + x$.

(B3) $x + 0 = x$.

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$.

(B6) $x \cdot 1 = x$.

(B7) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(B8) $x + (y \cdot z) = (x + y) \cdot (x + z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

Although this is an algebraic structure, we can also define a partial order by

$$y \leq x \Leftrightarrow y = y \cdot x.$$

An element x is called an *atom* if $y \leq x$ implies either that $y = x$ or $y = 0$.

A Boolean algebra is said to be *atomless* if it has no atoms.

A topological space X is called a *Boolean space* if it is compact, Hausdorff and 0-dimensional (that is, it has a base of clopen sets).

- If B is a Boolean algebra, then one can construct a Boolean space, X(B), called the *Stone space* of B.

- If X is a Boolean space, then one can construct a Boolean algebra, B(X), of the clopen subsets of X.

Theorem (Stone, 1937) Classical Stone duality.

1. *For any Boolean algebra* B*, we have that* $B \cong \mathsf{B}(\mathsf{X}(B))$*.*

2. *For any Boolean space* X*, we have that* $X \cong \mathsf{X}(\mathsf{B}(X))$*.*

Examples

1. If B is a finite Boolean algebra, then X(B) is a finite set regarded as a discrete topological space consisting of all the atoms of B.

2. If B is a (=the) countably infinite atomless Boolean algebra (which I call the *Tarski algebra*) then X(B) is the Cantor space.

Proofs There are two kinds of proof:

1. Use prime filters (=ultrafilters). See Givant and Halmos.

2. Use locales and then specialize. See Johnstone.

2. Ideas behind non-commutative Stone duality

1. Replace the Boolean algebra by some kind of semigroup which has a 'Boolean character'. The algebraic structure will be used to define a partial order. The $\cdot$ of the Boolean algebra will become the semigroup operation, and the $+$ will become a partially defined join. The 'Boolean character' will be reflected in the fact that a set of idempotents will be required to form a Boolean algebra.

2. Replace the topological space by a (1-sorted) topological category. This is a small category equipped with a topology with respect to which the operations $\mathbf{d}$ (dom), $\mathbf{r}$ (cod), $\mathbf{m}$ (multiplication) are required to be continuous. In addition, we require that the space of identities is open and forms a Boolean space.

3. Non-commutative Stone duality

The generalization of Boolean algebras.

We shall replace Boolean algebras by a class of monoids called Boolean right restriction monoids.

We shall motivate this abstract class of monoids with a concrete example.

Denote by $\mathsf{PT}(X)$ the set of all partial functions defined on the (non-empty) set X.

An element of $\mathsf{PT}(X)$ has the form $f \colon A \to X$ where $A \subseteq X$. We call the subset A the *domain of definition* of f denoted by $\mathrm{dom}(f)$.

Denote by f^* the identity function defined on $\mathrm{dom}(f)$. Whereas identity functions defined on subsets of X are idempotents, it is not true that all idempotents have this form. Idempotents that are identity functions defined on subsets are called *projections*. The set of projections of $\mathsf{PT}(X)$ is denoted by $\mathsf{Proj}(\mathsf{PT}(X))$.

Partial functions f and g can be compared using subset inclusion. We say that $f \subseteq g$ precisely when g restricted to the domain of definition of f is f itself. In fact, $f \subseteq g$ if and only if $f = gf^*$. Thus we do not have to impose a partial order from the outside since the order can be defined purely algebraically.

If $f, g \in \mathsf{PT}(X)$, it is not true in general that $f \cup g \in \mathsf{PT}(X)$ since $f \cup g$ might not be a partial function. It is a partial function precisely when for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ we have that $f(x) = g(x)$. This means precisely that $f(f^*g^*) = g(f^*g^*)$ which simplifies to $fg^* = gf^*$. We shall say that f and g are *left-compatible* if $fg^* = gf^*$. The partial functions f and g are left-compatible precisely when $f \cup g$ is also a partial function.

Observe that the set of projections is a commutative, idempotent subsemigroup. In fact, the set of projections forms a Boolean algebra since it is isomorphic to the powerset of X with respect to subset inclusion.

There is a map $f \mapsto f^*$ from $\mathsf{PT}(X)$ to $\mathsf{Proj}(\mathsf{PT}(X))$. Thus we regard $\mathsf{PT}(X)$ as an algebra of type $(2,1)$. Observe that $ff^* = f$ and that $f^*g = g(fg)^*$.

We now abstract the above example.

We define a monoid S to be a *right restriction monoid* if it is equipped with a unary operation $a \mapsto a^*$ satisfying the following axioms:

(RR1) $(s^*)^* = s^*$.

(RR2) $(s^*t^*)^* = s^*t^*$.

(RR3) $s^*t^* = t^*s^*$.

(RR4) $ss^* = s$.

(RR5) $(st)^* = (s^*t)^*$.

(RR6) $t^*s = s(ts)^*$.

The unary operation $s \mapsto s^*$ is called *star*. Denote by $\mathsf{Proj}(S)$ those elements a such that $a^* = a$ and call them *projections*.

We define a left restriction monoid, dually.

A *bi-restriction monoid* is a monoid which is both a left and right restriction monoid and the sets of projections are the same.

Let S be a right restriction monoid. Define

$$y \leq x \text{ iff } y = xy^*.$$

This is a partial order with respect to which the monoid is partially ordered. This is called the *natural partial order*.

If $a, b \leq c$ then $ab^* = ba^*$. More generally, we say that a and b are *left compatible* if $ab^* = ba^*$. Being left compatible is therefore a necessary condition to have a join.

A *Boolean right restriction monoid* is a right restriction monoid that satisfies the following three conditions:

1. Left compatible elements have joins.

2. Multiplication distributes over any joins that exist.

3. The set of projections forms a Boolean algebra with respect to the natural partial order.

The monoids PT(X) are Boolean right restriction monoids.

The generalization of Boolean spaces.

A topological category is said to be *domain-étale* if **d** is a local homeomorphism. Dually, we talk about *range-étale*, and *étale* if both.

A domain-étale catgeory is said to be *Boolean* if its space of identities is a Boolean space.

We now have all the ingredients we need for non-commutative Stone duality:

- Boolean restriction monoids.

- Boolean domain-étale topological caegories.

We sketch out how to construct a Boolean domain-étale topological category $\mathsf{C}(S)$ from a Boolean restriction monoid S.

Put $\mathsf{C}(S)$ equal to the prime filters in S.

If A is a prime filter, put

$$\mathbf{d}(A) = (A^*)^{\uparrow} \text{ and } \mathbf{r}(A) = \{e \in \mathsf{Proj}(S) \colon eA \subseteq A\}^{\uparrow}.$$

Both of these are also prime filters.

If A and B are prime filters and $\mathbf{d}(A) = \mathbf{r}(B)$ define

$$A \cdot B = (AB)^{\uparrow}.$$

This gives us a category on $\mathsf{C}(S)$.

If $a \in S$ let X_a be all the prime filters of S that contain a.

Put $\beta = \{X_a \colon a \in S\}$. This is the base for a topology on $\mathsf{C}(S)$.

Theorem *If S is a Boolean right restriction monoid then* $\mathsf{C}(S)$ *is a Boolean domain-étale topological category.*

We now go in the opposite direction.

Let C be a Boolean domain-étale topological category. A subset $A \subseteq C$ is called a *local section* if $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ then $a = b$.

Denote by $\mathsf{KS}(C)$ the set of all compact-open local sections of C.

Theorem *If C is a Boolean domain-étale topological category then* $\mathsf{KS}(C)$ *is a Boolean right restriction monoid.*

Theorem Non-commutative Stone duality.

1. *If S is a Boolean right restriction monoid then $S \cong$ KS(C(S)).*

2. *If C is a Boolean domain-étale topological category then $C \cong$ C(KS(C)).*

This was first proved by Cockett and Garner (2021) using locales; the proof I have sketched using prime filters is Lawson (2024).

4. The étale case

Let S be a Boolean right restriction monoid. We can ask the question when $\mathsf{C}(S)$ is a groupoid.

We say that $a \in S$ is a *partial unit* if there exists $b \in S$ such that $ab = b^*$ and $ba = a^*$.

The set of partial units is denoted by $\mathsf{Inv}(S)$ and is a Boolean inverse monoid.

We say that a Boolean right restriction monoid is *étale* if each element is a finite join of left-compatible partial units.

Theorem (Garner) *Let S be a Boolean right restriction monoid. Then* $\mathsf{C}(S)$ *is a groupoid iff S is étale.*

Theorem (Lawson, 2024) *The structure of an étale Boolean right restriction monoid S is completely determined by* $\mathsf{Inv}(S)$.

5. Other special cases

- Boolean inverse monoids and Boolean étale topological groupoids are imporant in the theory of C^*-algebras. Resende, 2007, Lawson, 2010, and Lawson & Lenz, 2013. Proved using prime filters.

- Boolean bi-restriction monoids and Boolean étale topological categories. Kudryavtseva & Lawson, 2017. Proved using locales.

6. Applications

- The duality between Boolean inverse monoids and Boolean groupoids is bound up with the theory of a class of groups, called *topological full groups*, that includes the Thompson-Higman groups. See the papers by Lawson, Sims & Vdovina, 2020, 2024.

- *Garner proved that the category of Cartesian closed universal algebras is equivalent to the category of Boolean right restriction monoids.*

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