4. MV-algebras and Boolean inverse monoids

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The goal of this talk is to show how two different generalizations of Boolean algebras are related to each other:

Boolean inverse monoids and MV-algebras.

MV stands for *multivalued logic*.

Terminology

I assume you are familiar with the contents of Lectures 1–3. In particular, you need to know what a Boolean inverse monoid is and what rook matrices are.

I shall use the following relations defined on any inverse semigroup:

Let e and f be idempotents. Then $e \mathscr{D} f$ if and only if there exists a such that $a^{-1}a = e$ and $aa^{-1} = f$.

$a \mathscr{J} b$ if and only if $SaS = SbS$. So, a and b generate the same principal ideal.

Let e and f be idempotents. Define $e \preceq f$ if and only if $e \mathscr{D} e' \leq f$ for some idempotent e'.

An inverse semigroup is said to be *Dedekind finite* if $e \mathscr{D} f \leq e$ implies that $e = f$.

In a Dedekind finite inverse semigroup, we have that $\mathscr{D} = \mathscr{J}$.

1. MV-algebras

MV-algebras are another generalization of Boolean algebras arising from multivalued logic.

See the book [CDM]: R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, 2000.

An *MV-algebra* is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and two constants 0 and 1 such that the following axioms hold:

1. $\oplus$ is associative.

2. $\oplus$ is commutative.

3. The identity is 0.

4. $\neg\neg x = x$.

5. $\neg 0 = 1$.

6. 1 is the zero.

7. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. This mysterious identity actually says that $x \vee y = y \vee x$. See below.

Examples

1. All Boolean algebras are MV-algebras.

2. The closed unit interval $[0, 1]$ is an MV-algebra when we define $\neg x = 1 - x$ and $x \oplus y = \max\{1, x + y\}$.

3. For each $n \geq n$, define
$$
\textsf{Ł}_n = \left\{ 0, \frac{1}{n-1}, \ldots, \frac{(n-2)}{(n-1)}, 1 \right\}
$$
with the above operations. Then $\textsf{Ł}_n$ is a finite MV-algebra.

An element x of an MV-algebra is called an *idempotent* if and only if $x \oplus x = x$.

The set of idempotents in an MV-algebra forms a Boolean algebra (Corollary 1.5.4 of [CDM]).

The following was proved as Corollary 1.5.5 of [CDM].

Theorem *An MV-algebra is a Boolean algebra if and only if every element is idempotent.*

Thus, MV-algebras are non-idempotent generalizations of Boolean algebras, though they are still commutative. There is a very rich theory of MV-algebras.

The following generalizes the structural description of finite Boolean algebras as finite direct products of $\mathbb{B}$, the two-element Boolean algebra.

It was proved as Proposition 3.6.5 of [CDM]).

Theorem *Every finite MV-algebra is a finite direct product of MV-algebras of the form Ł$_n$.*

Define $x \le y$ if and only if $x \oplus z = y$ for some z. Then $\le$ is a partial order (Lemma 1.1.2 of [CDM]). The folllowing was proved as Proposition 1.1.5 and Proposition 1.5.1 of [CDM].

Proposition *Every MV-algebra is a distributive lattice with respect to the partial order $\le$.*

It can be proved that

$$x \vee y = \neg(\neg x \oplus y) \oplus y$$

and

$$x \wedge y = \neg(\neg x \vee \neg y)$$

by Proposition 1.1.5 of [CDM].

The following will be useful later. It can also be proved that

$$x \oplus y = x \oplus (\neg x \wedge y)$$

where $x \le \neg(\neg x \wedge y)$.

Boolean inverse monoids generalize Boolean algebras.

MV-algebras generalize Boolean algebras.

This raises the question of how Boolean inverse monoids and MV-algebras are related.

We shall show that the theory of MV-algebras is subservient to the theory of Boolean inverse monoids.

We shall refer to our paper

[LS]: M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *Journal of Pure and Applied Algebra* **221** (2017), 45–74.

See also [W]: F. Wehrung, *Refinement monoids, equidivisibility types, and Boolean inverse semigroups*, Springer, lecture Notes in Mathematics 2188, 2017.

2. The type monoid

How can we study Boolean inverse monoids?

We need invariants.

The first such invariant was introduced in

[KLLR]: G. Kudryavtseva, M. V. Lawson, D. H. Lenz, P. Resende, Invariant means on Boolean inverse monoids, *Semigroup Forum* **92** (2016), 77–101.

Other invariants are the homology groups associated with the Stone groupoid of the Boolean inverse monoid.

Let S be a Boolean inverse semigroup (you can think monoid but there are technical reasons to allow semigroups).

Denote the idempotent $\mathscr{D}$-class containing the idempotent e by $[e]$. Define $\mathsf{Int}(S)$ to be the set $\frac{\mathsf{E}(S)}{\mathscr{D}}$ equipped with the following *partially defined* operation

$$[e] + [f] = [e' \vee f']$$

if $e \mathrel{\mathscr{D}} e'$ and $f \mathrel{\mathscr{D}} f'$ and $e'f' = 0$ and undefined otherwise. We write $\exists [e] + [f]$ if the sum is defined.

We call $(\mathsf{Int}(S), +, [0])$ the *interval* associated with S.

A structure $(E, +, 0)$ is called a *partial commutative monoid* if it satisfies the axioms (E1), (E2) and (E3) below:

(E1) $\exists a + b$ if and only if $\exists b + a$ and when they exist they are equal.

(E2) $\exists (a + b) + c$ if and only if $\exists a + (b + c)$ and when they exist they are equal.

(E3) For all $a \in E$, $\exists a + 0$ and $a + 0 = a$.

(E4) There is an element 1 such that $\exists a + 1$ implies that $a = 0$.

We say that a partial commutative monoid $(E, +, 0)$ is *conical* if $a + b = 0$ implies that $a = 0$ and $b = 0$.

It is said to satisfy the *refinement property* if $a_1 + a_2 = b_1 + b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$ and $a_2 = c_{21} + c_{22}$.

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Proposition *Let S be a Boolean inverse semigroup. Then $(\mathsf{Int}(S), +, [0])$ is a conical partial commutative monoid satisfying the refinement property. If S is also a monoid then it satisfies axiom (E4) where we use [1].*

Put $\mathbf{0} = [0]$ and $\mathbf{1} = [1]$, in the monoid case.

This construction is actually functorial.

The problem with $(\mathsf{Int}(S), +, [0])$ is that the addition is partially defined.

We can embed $\mathsf{Int}(S)$ into an honest-to-goodness commutative monoid by using rook matrices.

Let S be a Boolean inverse monoid. Then $M_\omega(S)$ is a Boolean inverse semigroup. This has the additional property that the addition in $\mathsf{Int}(M_\omega(S))$ is always defined.

Define the *type monoid* of S, denoted by $\mathsf{Typ}(S)$, to be the commutative monoid $\mathsf{Int}(M_\omega(S))$. Observe that $\mathsf{Int}(S)$ can be embedded into $\mathsf{Typ}(S)$, and $\mathsf{Typ}(S)$ has the correct universal properties.

A partial commutative monoid $(E, +, ', 0, 1)$ satisfying (E4) is called an *effect algebra* if it also has a unary operation $'$ such that (E5) holds: for each $a \in E$ we have that $\exists a + a'$ and $a + a' = 1$.

It is natural to ask when the interval of S is an effect algebra.

Effect algebras were developed independently by mathematical physicists studying quantum measurement theory and quantum effects.

An inverse monoid is said to be *factorizable* if each element is below an element in the group of units.

The following result shows how the inverse monoid structure and the Boolean algebra structure interact.

Lemma [Proposition 2.7 of [LS]] *Let S be a Boolean inverse monoid. Then S is factorizable if and only if $e \mathcal{D} f$ implies that $\bar{e} \mathcal{D} \bar{f}$.*

Proposition *Let S be a Boolean inverse monoid. Then its interval is an effect algebra if and only if S is factorizable where we define $[e]' = [\bar{e}]$.*

In a partial commutative monoid M, it is natural to define a preorder $a \leq b$ if $a + c = b$ for some c.

Lemma *Let S be a Boolean inverse monoid. Then $[e] \leq [f]$ if and only if $e \preceq f$ in S. If S is factorizable then $\leq$ is a partial order.*

3. Foulis inverse monoids

A *Foulis inverse monoid* is defined to be a factorizable Boolean inverse monoid S such that $S/\mathscr{J} = S/\mathscr{D}$ is a lattice under subset inclusion.

If S is a Foulis monoid, put $\mathsf{Ł}(S) = \mathsf{Int}(S)$ equipped with the following everywhere defined operation $\oplus$:

$$[e] \oplus [f] = [e] + (\neg[e] \wedge [f])$$

where $[e] \wedge [f]$ is constructed using the lattice structure on $S/\mathscr{J}$.

The following is a nice introduction to our theory:

[WL]: Weiyun Lu, *Topics in many-valued and quantum algebraic logic*, MSc Thesis, University of Ottawa, 2016.

Theorem [Section 2.2 of [LS]] *Let S be a Foulis monoid. Then $(Ł(S), \oplus, \neg, \mathbf{0}, \mathbf{1})$ is an MV-algebra.*

An MV-algebra isomorphic to one of the form $Ł(S)$ where S is a Foulis monoid is said to be *coordinatizable*. WLOG we can always assume that S is fundamental.

Lemma *Let S be a Foulis monoid. Then $Ł(S)$ is a Boolean algebra if and only if all idempotents in S are central.*

Finite symmetric inverse monoids are Foulis monoids as are matricial inverse monoids.

Recall that a Boolean inverse monoid is said to be *matricial* if it is isomorphic to a finite direct product of finite symmetric inverse monoids.

Theorem [Theorem 2.14 of [LS]] *The finite MV-algebra* $Ł_{n+1}$ *is coordinatized by* $\mathcal{I}_n$. *More generally, every finite MV-algebra is coordinatized by a matricial monoid.*

The following was first proved in [LS].

Theorem *Every countable MV-algebra is coordinatizable.*

The above theorem was generalized by Wehrung [W] as his Theorem 5.2.10 using different techniques.

Theorem *Every MV-algebra is coordinatizable.*

The above theorem therefore completely answers the question of the nature of the relationship between MV-algebras and Boolean inverse monoids.

4. AF inverse monoids

We shall sketch out our result on the coordinatization of countable MV-algebras.

We first need to be explicit about what kind of substructures we shall be dealing with.

Let S be a Boolean inverse monoid. Let T be an inverse submonoid of S which is a Boolean inverse monoid in its own right. We say that T is a *subalgebra* of S if $E(T)$ is a Boolean subalgebra of $E(S)$ and compatible joins in T are the same as compatible joins in S.

Let S be a countable Boolean inverse monoid. We say it is *AF (Approximately Finite)* if $S = \bigcup_{i=0}^{\infty} S_i$ where $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and where each S_i is a matricial subalgebra of S. We assume that $S_0 = \mathbb{B}$. They are always factorizable.

This definition is modelled after the definition of *AF C^*-algebras* where finite direct products of finite symmetric inverse monoids replace finite dimensional C^*-algebras.

The theory of AF C^*-algebras was introduced in:

[B]: O. Bratteli, Inductive limits of finite dimensional C^*-algebras, *Transactions of the AMS* **171** (1972), 195–234.

The theory developed there relies a lot on matrix units which form inverse semigroups.

Bratteli diagrams give rise to AF inverse monoids and conversely.

In [DM]: D. Mundici, Logic of infinite quantum systems, *International Journal of Theoretical Physics* **32** (1993), 1941–1955, we have the following quote:

> . . . AF C^*algebras should be regarded as sort of noncommutative Boolean algebras . . .

We claim that the above quote actually applies more strongly to AF inverse monoids since commutative AF inverse monoids *are* Boolean algebras.

Theorem 3 from [DM] states that there is a bijection between AF C^*-algebras whose Murray-von Neumann order is a lattice and countable MV-algebras.

By modifying Mundici's proof we actually proved the following:

Theorem [LS] *Every countable MV-algebra can be coordinatized by an AF inverse monoid satisfying the lattice condition.*

Research Question 3: Describe the Boolean groupoids G(S) when S is a Foulis monoid.

Research Question 4: Use the (extensive) theory of MV-algebras to reveal something about the structure of Foulis monoids.

END OF LECTURE 4