2. Boolean inverse monoids

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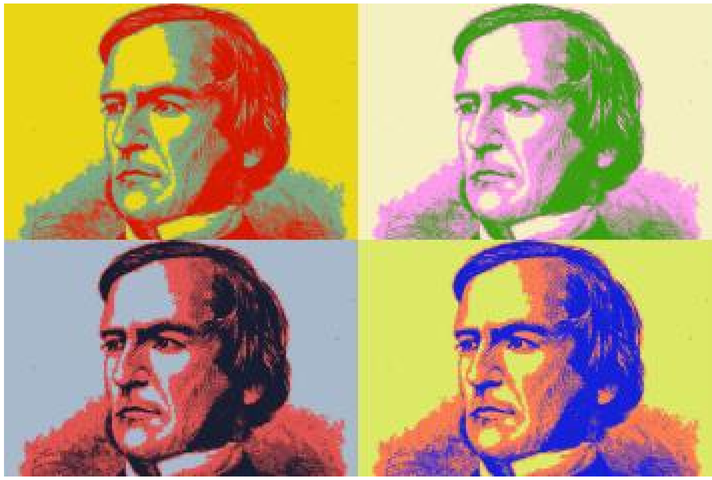

I shall assume familiarity with Lecture 1.

Recall that in an inverse semigroup S the elements a and b having an upper bound implies that a and b are compatible.

Define a and b to be *compatible*, denoted by $a \sim b$, if and only if $a^{-1}b$ and ab^{-1} are idempotents. Being compatible is therefore a necessary condition for a pair of elements to have a join.

1. Basic definitions

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

Pseudogroups are the correct abstractions of pseudogroups of transformations.

This leads us to think of inverse semigroup theory from a lattice-theoretic perspective.

An inverse semigroup is a *meet-semigroup* if has has all binary meets.

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Summary

Commutative	**Non-commutative**
Meet semilattice	Inverse semigroup
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet-semigroup

2. Motivation

This work draws on a number of sources:

- The idea of a *non-commutative space*.
- The 1980 book by Renault on the relationship between topological groupoids and C^*-algebras.
- Paterson's 1999 book on groupoids, inverse semigroups and operator algebras.
- Work in the late 90's by Johannes Kellendonk on tiling semigroups as mediated by Daniel Lenz.
- Birget's semigroup approach to constructing the Thompson groups.
- The work of Charles Ehresmann.
- Frame theory as described in Peter Johnstone's book.
- The quantale theory of Pedro Resende.

In concrete terms, this leads us to regard inverse semigroups (of various complexions) as being non-commutative generalizations of lattices (of various complexions).

There are various applications of this approach:

1. Topos theory and pseudogroups.
2. Constructing Thompson-Higman type groups.
3. Constructing C^*-algebras from inverse semigroups.

In what follows, I shall focus on Boolean inverse monoids.

3. Generalities on Boolean inverse monoids

An inverse monoid is *Boolean* if it satisfies the following three conditions:

1. Compatible pairs of elements have all binary compatible joins: we write $a \vee b$ for the join of a and b.

2. Multiplication distributes over such joins: so $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$.

3. The set of idempotents $\mathsf{E}(S)$ of S forms a Boolean algebra with respect to the usual order.

It is useful to define the *extent* of a, denoted by $\mathbf{e}(a)$, to be $\mathbf{d}(a) \vee \mathbf{r}(a)$.

We say a and b are *orthogonal*, denoted by $a \perp b$, if $\mathbf{d}(a)\mathbf{d}(b) = 0$ and $\mathbf{r}(a)\mathbf{r}(b) = 0$. In terms of partial bijections, this means that domains and ranges are disjoint.

Orthogonal elements are compatible. We sometimes denote orthogonal joins by $\oplus$.

The complement of e in a Boolean algebra is denoted by $\bar{e}$.

The two-element Boolean algebra is denoted by $\mathbb{B}$.

If $a \leq b$ in a Boolean inverse monoid define $b \setminus a = b\overline{\mathbf{d}(a)}$. Observe that $b = a \oplus (b \setminus a)$.

Let S and T be Boolean inverse monoids.

A *morphism* $\theta \colon S \to T$ is a semigroup homomorphism that preserves identities and binary compatible joins.

Boolean inverse monoids behave a lot like rings.

The join should be viewed as a *partially defined* addition.

We shall see later that we can form matrices over Boolean inverse monoids.

4. Examples of Boolean inverse monoids

Any group with an adjoined zero. We write these as G^0.

The symmetric inverse monoids $\mathcal{I}(X)$ are Boolean inverse monoids (see Lecture 1).

Boolean inverse monoids arise naturally as soon as you are interested in embedding inverse monoids in rings.

Theorem [Lawson-Paterson] *Let S be an inverse submonoid with zero of the multiplicative monoid of a ring R. Then there is a Boolean inverse submonoid S'' such that $S \subseteq S'' \subseteq R$.*

Every inverse monoid with zero gives rise to a Boolean inverse monoid.

Theorem [Booleanization/Lawson] *Let* S *be an inverse monoid with zero. Then there is a Boolean inverse monoid* $\mathsf{B}(S)$ *and an embedding* $\beta \colon S \to \mathsf{B}(S)$ *such that if* $\theta \colon S \to T$ *is an homomorphism to a Boolean inverse monoid then there is a unique morphism* $\gamma \colon \mathsf{B}(S) \to T$ *such that* $\theta = \beta\gamma$*.*

Let S be a Boolean inverse monoid. An $n \times n$ *rook matrix* over S is an $n \times n$ matrix whose entries are drawn from S with only a finite number of non-zero entries such that if a and b are in the same row then $\mathbf{r}(a) \perp \mathbf{r}(b)$ and if they are in the same column then $\mathbf{d}(a) \perp \mathbf{d}(b)$. Denote the set of all $n \times n$ rook matrices over S by $M_n(S)$. If n is countably infinite we write $M_\omega(S)$; we shall use this case in Lecture 4.

Lemma *When n is finite, $M_n(S)$ is a Boolean inverse monoid for matrix multiplication when sums are replaced by joins. The inverse of a rook matrix A is denoted by A^* and is the transpose of A with all entries inversed. The idempotents are the diagonal rook matrices whose diagonal entries are idempotents. $M_\omega(S)$ is a Boolean semigroup rather than a Boolean monoid.*

Our use of the term 'rook matrix' generalizes that of Solomon. Here, 'rook' is the alternative name for a 'castle' in chess.

5. Additive ideals in Boolean inverse monoids

A non-empty subset $I \subseteq S$ is called an *additive ideal* if it is a semigroup ideal closed under binary compatible joins. We call $\{0\}$ the *trivial additive ideal*.

Let $\theta \colon S \to T$ be a morphism between Boolean inverse monoids. Define the *kernel of* θ, denoted by $\ker(\theta)$, to be all the elements of S that map to zero. Kernels are additive ideals.

Although Boolean inverse monoids behave like rings in many ways, one has to be careful with morphisms. The following result is a warning.

Lemma *Let* $\theta \colon S \to T$ *be a morphism between Boolean inverse monoids. Then the kernel of* θ *is trivial if and only if* θ *is idempotent-separating.*

If I is an additive ideal on S, then we can define a congruence $\equiv_I$ on S by $a \sim_I b$ if and only if there is $c \leq a, b$ such that $a \setminus c, b \setminus c \in I$. Then $S/I = S/\sim_I$ is a Boolean inverse monoid and the natural map is a morphism.

Not all morphisms are induced by additive ideals.

Lemma *A morphism $\theta \colon S \to T$ is induced by an additive ideal if and only if it is* weakly-meet-preserving *meaning that if $t \leq \theta(a), \theta(b)$ then there exists $c \leq a, b$ such that $t \leq \theta(c)$.*

We say that a Boolean inverse monoid is 0-*simplifying* if the only additive ideals are $\{0\}$ and S, itself.

We say that a Boolean inverse monoid is *simple* if it is 0-simplifying and fundamental. This is justified by the following lemma.

Lemma *Let S be a simple Boolean inverse monoid. If $\theta \colon S \to T$ is a surjective morphism then θ is an isomorphism.*

An inverse semigroup with zero is said to be 0-*simple* if the only ideals are $\{0\}$ and S itself. A Boolean inverse monoid is *congruence-free* if it is 0-simple and fundamental.

6. Infinitesimals in Boolean inverse monoids

A typical element of $\mathcal{I}_n$ can be written as a join (in fact, an orthogonal join) of elements of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ and elements of the form $\begin{pmatrix} x \\ x \end{pmatrix}$. The latter are idempotents, whereas the former are examples of the following concept.

A non-zero element a in a Boolean inverse semigroup is called an *infinitesimal* if $a^2 = 0$.

Infinitesimals are important because they can be used to build units.

Lemma *Let a be an infinitesimal in a Boolean inverse monoid. Then $g = a \vee a^{-1} \vee \overline{\mathbf{e}(a)}$ is an involution.*

The Boolean inverse monoids that do not contain infinitesimals are very special.

Lemma *A Boolean inverse monoid has no infinitesimals if and only if all its idempotents are central.*

Lemma *A Boolean inverse monoid is fundamental if and only if every non-idempotent is above an infinitesimal.*

We say that a Boolean inverse monoid is *basic* if every element is a join of a finite number of infinitesimals and an idempotent. Finite direct products of finite symmetric inverse monoids are basic.

A Boolean inverse monoid is said to be *piecewise factorizable* if each element has the form $\bigvee_{i=1}^{m} g_i e_i$ where the g_i are units and the e_i are idempotents.

Lemma *Basic Boolean inverse monoids are fundamental meet-monoids which are piecewise factorizable.*

7. Finite Boolean inverse monoids

It is a feature of semigroup theory that the finite members of a class are usually hard to describe.

This is not true of finite Boolean inverse monoids.

Their theory can be developed in an entirely elementary way by generalizing the classical structure theory of finite Boolean algebras.

Recall that an *atom* in a partially ordered set (with zero) is an element a such that $b \leq a$ implies that $b = 0$ or $b = a$.

Theorem *Let S be a finite Boolean inverse monoid.*

1. *The set of atoms G of S forms a groupoid under the restricted product and $S \cong \mathsf{K}(G)$, the local bisections of G.*

2. *S is fundamental if and only if G is a principal groupoid.*

3. *S is 0-simplifying if and only if G is a connected groupoid.*

Theorem *Let S be a finite Boolean inverse monoid.*

1. *S is isomorphic to a finite direct product $S_1 \times \ldots \times S_m$ where each S_i is a 0-simplifying Boolean inverse monoid.*

2. *S is fundamental if and only if $S \cong \mathcal{I}_{n_1} \times \ldots \times \mathcal{I}_{n_m}$ for some $n_1, \ldots, n_m$. We call such Boolean inverse monoids* matricial.

3. *S is simple if and only if $S \cong \mathcal{I}_n$ for some n.*

Two insights emerge from these results.

The groups of units of the finite simple Boolean inverse monoids are the finite symmetric groups. We therefore regard the groups of units of the infinite simple Boolean inverse monoids as infinite generalizations of the finite symmetric groups.

The matricial Boolean inverse monoids are analogous to the finite-dimensional C^*-algebras: where finite replaces finite-dimensional and $\mathbb{B}$ replaces $\mathbb{C}$.

We can use the rook matrices introduced earlier to describe finite Boolean inverse monoids.

Theorem *Let S be a finite Boolean inverse monoid.*

1. *S is isomorphic to a finite direct product*

$$M_{n_1}(G_1^0) \times \ldots \times M_{n_m}(G_m^0)$$

   where $G_1, \ldots, G_m$ are finite groups, and if G is a group then G^0 is that group with an adjoined zero.

2. *S is fundamental if and only if*

$$S \cong M_{n_1}(\mathbb{B}) \times \ldots \times M_{n_m}(\mathbb{B}),$$

   where $\mathbb{B}$ is the two-element Boolean algebra, and for some $n_1, \ldots, n_m$.

3. *S is simple if and only if $S \cong M_n(\mathbb{B})$ for some n.*

8. The dichotomy theorem for Boolean inverse monoids

A Boolean inverse monoid is said to be *atomless* if it contains no atoms.

We call the following the Dichotomy Theorem.

Theorem *Let S be a simple Boolean inverse monoid. Then there are exactly two possibilities:*

1. $S \cong \mathcal{I}_n$ *for some* n.

2. S *is atomless.*

There is exactly one countable atomless Boolean algebra which we call the *Tarski algebra*.

Let S be a Boolean inverse monoid. An idempotent e is said to be *properly infinite* if there are elements x and y such that $\mathbf{d}(x) = e = \mathbf{d}(y)$, $\mathbf{r}(x), \mathbf{r}(y) \leq e$ and $\mathbf{r}(x) \perp \mathbf{r}(y)$.

A Boolean inverse monoid is said to be *purely infinite* if every non-zero idempotent is properly infinite. The following is simply the inverse monoid version of Proposition 4.11 of Matui:

[M]: H. Matui, Topological full groups of one-sided shifts of finite type, *J. Reine Angew. Math.* **705** (2015), 35–84.

Lemma *Let S be an atomless Boolean inverse monoid. Then it is 0-simple if and only if it is 0-simplifying and purely infinite.*

The following is Theorem 4.16 of [M] translated into our language.

Theorem *Let S be an atomless Boolean inverse monoid. If it is congruence-free then the derived subgroup of its group of units of S is simple.*

How to translate from the language of étale groupoids (used by Matui) into the language of Boolean inverse monoids is described in the next lecture.

The following are now interesting questions:

Research Question 1: Classify the countably infinite simple Boolean inverse monoids.

Research Question 2: Describe the groups of units of the countably infinite simple Boolean inverse monoids. Classical groups often arise as groups of units of rings. The Classical Thompson groups V_n arise as the groups of units of countably infinite simple Boolean inverse monoids.

Books and papers

The theory of Boolean algebras is described in the following book

S. Givant, P. Halmos, *Introduction to Boolean algebras*, Springer, 2009.

Boolean inverse monoids were introduced in the following paper

M. V. Lawson, A noncommutative generalization of Stone duality, *J. Aust. Math. Soc.* **88** (2010), 385–404.

A good account of the theory of Boolean inverse monoids can be found in

F. Wehrung, *Refinement monoids, equidivisibility types, and Boolean inverse semigroups*, Springer, Lecture Notes in mathematics 2188, 2017.

The following does what its title says it does

Mark V. Lawson, Recent developments in inverse semigroup theory, *Semigroup Forum* **100** (2020), 103–118.

END OF LECTURE 2