2. ETALE GROUPOIDS

MARK V. LAWSON

ABSTRACT. In this article, we define étale groupoids and describe some of their properties.

1. Generalities

1.1. **Categories.** A category is usually regarded as a 'category of structures' of some kind, such as the category of sets or the category of groups. A (small) category can, however, also be regarded as an algebraic structure; that is, as a set equipped with a partially defined binary operation satisfying certain axioms. We shall need both perspectives but in this article the latter perspective will be foremost. This algebraic approach to categories was an important ingredient in Ehresmann's work [1] and applied by Philip Higgins to prove some basic results in group theory [3]. In addition, groupoids have come to play an important role in constructing C^*-algebras [5, 6].

To define the algebraic notion of a category, we begin with a set C equipped with a partial binary operation. We write $\exists ab$ to mean that the product ab is defined. An *identity* in such a structure is an element e such that if $\exists ae$ then $ae = a$ and if $\exists ea$ then $ea = a$. A *category* is a set equipped with a partial binary operation satisfying the following axioms:

(C1): $\exists a(bc)$ if and only if $\exists (ab)c$ and when one is defined so is the other and they are equal.

(C2): $\exists abc$ if and only if $\exists ab$ and $\exists bc$.

(C3): For each a there is an identity e, perforce unique, such that $\exists ae$, and there exists an identity f, perforce unique, such that $\exists fa$. I shall write $\mathbf{d}(a) = e$ and $\mathbf{r}(a) = f$ and draw the picture

$$f \xleftarrow{a} e.$$

The set of all elements from e to f is called a *hom-set* and denoted $\hom(e, f)$.

You can check that $\exists ab$ if and only if $\mathbf{d}(a) = \mathbf{r}(b)$.

Example 1.1. A category with one identity is a monoid. Thus, viewed in this light, categories are 'monoids with many identities'.

1.2. **Groupoids.** Before we define groupoids, we define inverse semigroups whose theory shadows that of groupoids. These semigroups will be the subject of a detailed account in the third article. A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique element s^{-1} such that $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$.

Example 1.2. The first example of an inverse semigroup is $I(X)$, the *symmetric inverse monoid* on the set X. This consists of all bijections between subsets of X with composition being composition of partial functions.

An element a of a category is said to be *invertible* if there exists an element b such that ab and ba are identities. If such an element b exists it is unique and is called the *inverse* of a; we denote the inverse of a when it exists by a^{-1}. A category in which every element is invertible is called a *groupoid*.

Example 1.3. A groupoid with one identity is a group. Thus groupoids are 'groups with many identities'.

Example 1.4. A set can be regarded as a groupoid in which every element is an identity.

Example 1.5. Equivalence relations can be regarded as groupoids. They correspond to *principal groupoids*; that is, those groupoids in which given any identities e and f there is at most one element g of the groupoid such that $f \xleftarrow{a} e$. A special case of such groupoids are the *pair groupoids*, $X \times X$, which correspond to equivalence relations having exactly one equivalence class.

Example 1.6. Let $G \times X \to X$ be a left group action of the group G on the set X. We can construct a groupoid $G \ltimes X$, called a *transformation groupoid*,

We shall need the following notation for the maps involved in defining a groupoid (not entirely standard). Define $\mathbf{d} = g^{-1}g$ and $\mathbf{r}(g) = gg^{-1}$. Put

$$G * G = \{(g,h) \in G \times G \colon \mathbf{d}(g) = \mathbf{r}(h)\}$$

and if $U, V \subseteq G$, define $U * V = (U \times V) \cap (G * G)$. Define $\mathbf{m} \colon G * G \to G$ by $(g,h) \mapsto gh$ and $\mathbf{i} \colon G \to G$ by $g \mapsto g^{-1}$. The set of identities of G is denoted by G_o. If e is an identity in G then G_e is the set of all elements a such that $a^{-1}a = e = aa^{-1}$. We call this the *local group at* e. Put $\mathrm{Iso}(G) = \bigcup_{e \in G_o} G_e$. This is called the *isotropy groupoid* of G.

We now show how to construct all groupoids. Let G be a groupoid. We say that elements $g, h \in G$ are *connected*, denoted $g \equiv h$, if there is an element $x \in G$ such that $\mathbf{d}(x) = \mathbf{d}(h)$ and $\mathbf{r}(x) = \mathbf{d}(g)$. The $\equiv$-equivalence classes are called the *connected components of the groupoid*. If $\exists gh$ then necessarily $g \equiv h$. It follows that $G = \coprod_{i \in I} G_i$ where the G_i are the connected components of G.

It remains to describe the structure of all connected groupoids. Let X be a non-empty set and let H be a group. The set of triples $X \times H \times X$ becomes a groupoid when we define $(x,h,x')(x',h',x'') = (x,hh',x'')$ and $(x,h,y)^{-1} = (y,h^{-1},x)$. It is easy to check that $X \times H \times X$ is a connected groupoid. Now let G be an arbitrary groupoid. Choose, and fix, an identity e in G. Denote the local group at e by H. For each identity f in G choose an element x_f such that $\mathbf{d}(x_f) = e$ and $\mathbf{r}(x_f) = f$. Put $X = \{x_f \colon f \in G_o\}$. We prove that G is isomorphic to $X \times H \times X$. Let $g \in G$. Then $x_{\mathbf{r}(g)}^{-1} g x_{\mathbf{d}(g)} \in H$. Define a map from G to $X \times H \times X$ by $g \mapsto (x_{\mathbf{r}(g)}, x_{\mathbf{r}(g)}^{-1} g x_{\mathbf{d}(g)}, x_{\mathbf{d}(g)})$. It is easy to show that this is a bijective functor.

1.3. **Partial bisections on groupoids.** The key definition needed to relate groupoids and inverse semigroups in our non-commutative generalization of Stone duality is the following. A subset $A \subseteq G$ is called a *partial bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Lemma 1.7. *A subset $A \subseteq G$ is a partial bisection if and only if $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ implies that $a = b$ and $\mathbf{r}(a) = \mathbf{r}(b)$ implies that $a = b$.*

Proof. Suppose that A is a partial bijection. Let $a, b \in A$ such that $\mathbf{d}(a) = \mathbf{d}(b)$. Then the product ab^{-1} exists and, by assumption, is an identity. It follows that $a = b$. A similar argument shows that if $a, b \in A$ are such that $\mathbf{r}(a) = \mathbf{r}(b)$ then $a = b$. We now prove the converse. We prove that $A^{-1}A \subseteq G_o$. Let $a, b \in A$ and suppose that $a^{-1}b$ is exists. Then $\mathbf{r}(a) = \mathbf{r}(b)$. By assumption, $a = b$ and so $a^{-1}b$ is an identity. The proof that $AA^{-1} \subseteq G_o$ is similar. $\square$

The next result tells us how to construct an inverse monoid from a groupoid.

Proposition 1.8. *The set of all partial bisections of a groupoid forms an inverse monoid under subset multiplication.*

Proof. Let A and B be partial bisections. We prove that AB is a partial bisection. We calculate $(AB)^{-1}AB$. This is equal to $B^{-1}A^{-1}AB$. Now $A^{-1}A$ is a set of identities. Thus $B^{-1}A^{-1}AB \subseteq B^{-1}B$. But $B^{-1}B$ is a set of identities. It follows that $(AB)^{-1}AB$ is a set of identities. By a similar argument we deduce that $AB(AB)^{-1}$ is a set of identities. We have therefore proved that the product of two partial bisections is a partial bisection. Since G_o is a partial bisection, we have proved that the set of partial bisections is a monoid. Observe that if A is a partial bisection, then $A = AA^{-1}A$ and $A = A^{-1}AA^{-1}$. Suppose that $A^2 = A$. Then $a = bc$ where $b, c \in A$. But $\mathbf{d}(a) = \mathbf{d}(c)$, and so $a = c$, and $\mathbf{r}(a) = \mathbf{r}(b)$, and so $a = b$. It follows that $a = a^2$. But the only idempotents in groupoids are identities and so a is an identity. We have show that if $A^2 = A$ then $A \subseteq G_o$. It is clear that if $A \subseteq G_o$ then $A^2 = A$. We have therefore proved that the idempotent partial bisections are precisely the subsets of the set of identities. Suppose that $A = ACA$ and $C = CAC$. We prove that $C = A^{-1}$. Observe that AC and CA are idempotents. Thus both are subsets of the set of identities. Let $c \in C$. Then $c = c'ac''$ for some $c', c'' \in C$ and $a \in A$. Now $\mathbf{d}(c) = \mathbf{c}''$. Thus $c = c''$. Similarly, $c = c'$. Thus $c = cac$. It follows that $c = a^{-1}$ and so $C \subseteq A^{-1}$. Let $a \in A$. Then by a similar argument to the above, $a = aca$ for some $c \in C$. It follows that $a^{-1} = c$. This prove that $A^{-1} \subseteq C$. We have therefore proved that $C = A^{-1}$, and so proved the claim. □

A subset $A \subseteq G$ of a groupoid is called a *bisection* if

$$A^{-1}A, AA^{-1} = G_o.$$

The following is immediate by Proposition 1.8 and tells us that we may also construct groups from groupoids.

Corollary 1.9. *The set of bisections is just the group of units of the inverse monoid of all partial bisections.*

The following example is important in motivating the theory of *topological full groups* of étale groupoids.

Example 1.10. The inverse monoid of partial bisections of the pair groupoid $X \times X$ is isomorphic with the symmetric inverse monoid on X, and the group of bisections of the pair groupoid $X \times X$ is isomorphic with the symmetric group on X. It is straightforward to check that there is a bijection between the partial bisections in the groupoid $X \times X$ and the set of partial bijections on the set X which induces a bijection between the set of bisections in the groupoid $X \times X$ and the set of bijections on the set X. It can be checked that composition of partial bisections agrees with composition of bijections.

2. Etale topological groupoids

This section is based on [2, 5, 7].

Just as we can study topological groups, so we can study topological groupoids. A *topological groupoid* is a groupoid G equipped with a topology, and G_o is equipped with the subspace topology, such that the maps $\mathbf{d}, \mathbf{r}, \mathbf{i}, \mathbf{m}$ are all continuous functions where $\mathbf{d}, \mathbf{r} \colon G \to G_o$ and $\mathbf{m} \colon G * G \to G$. Clearly, it is just enough to require that $\mathbf{m}$ and $\mathbf{i}$ are continuous. Observe that $\mathbf{i}$ is actually a homeomorphism.

A topological groupoid is said to be *open* if the map $\mathbf{d}$ is an open map; it is said to be *étale* if the map $\mathbf{d}$ is a local homeomorphism. Recall that every local homeomorphism is an open map and so étale groupoids are open groupoids.

Lemma 2.1. *Let G be a topological space. If G_o is an open subset then the open partial bisections form a basis for the topology.*

Proof. Since $\mathbf{m}$ is continuous, $\mathbf{m}^{-1}(G_o)$ is an open subset of $G * G$. It follows that $\mathbf{m}^{-1}(G_o) = \left(\bigcup_{i \in I} U_i \times V_i\right) \cap (G * G)$, for some open sets $U_i, V_i \subseteq G$ where $i \in I$, using the description of the product topology. Observe that $U_i V_i \subseteq G_o$ for each $i \in I$. Let $g \in G$. Then $(g^{-1}, g) \in G * G$ and $\mathbf{m}(g^{-1}, g) = g^{-1}g \in G_o$. Thus $(g^{-1}, g) \in U_i \times V_i$ for some $i \in I$. Put $X = U_i^{-1} \cap V_i$, an open set containing g. Then $X^{-1}X \subseteq G_o$. By a symmetric argument, we can find an open set Y containing g such that $YY^{-1} \subseteq G_o$. Put $A = X \cap Y$. Then A is an open set that contains g and is a partial bisection by construction. □

Lemma 2.2. *Let G be a topological groupoid. If $\mathbf{d}\colon G \to G_o$ is a local homeomorphism then G_o is an open subset of G.*

Proof. Let $e \in G_o$. Then, of course, $e \in G$. Since $\mathbf{d}$ is a local homeomorphism, there is an open set $e \in A$ in G and an open set $e \in B$ in G_o such that $(\mathbf{d}|A)\colon A \to B$ is a homeomorphism. Put $B' = A \cap B$. Then $e \in B'$ and B' is an open subset of B. It follows that $A' = (\mathbf{d}|A)^{-1}(B') = \mathbf{d}^{-1}(B') \cap A$ is an open set in G. It is easy to check that $B' \subseteq A'$. But $\mathbf{d}$ is the identity function on B' and $\mathbf{d}$ restricts to a bijective map from A' to B'. It follows that $A' = B'$ and so $e \in A' \subseteq G_o$, where A' is open in G. It follows that G_o is an open subset of G. □

Lemma 2.3. *Let G be a topological groupoid. If $\mathbf{d}\colon G \to G_o$ is a local homeomorphism then $\mathbf{m}$ is an open map.*

Proof. By Lemma 2.2 and Lemma 2.1, we know that the open partial bisections form a basis for the topology. Since $\mathbf{d}$ is a local homeomorphism so too is $\mathbf{r}$ using the fact that $\mathbf{i}$ is a homeomorphism. Let $A, B \subseteq G$ be open sets. Then $A * B = (A \times B) \cap (G * G)$ is an open set. We prove that $\mathbf{m}(A * B) = AB$ is also open from which it follows that $\mathbf{m}$ is an open map. Let $(u, v) \in A * B$. Then $uv \in AB$. There is an open partial bisection W such that $uv \in W$. It follows that there is an open neighbourhood Z of $\mathbf{r}(uv)$ such that $(\mathbf{r}|W)\colon W \to Z$ is a homeomorphism. From the fact that $\mathbf{m}$ is continuous, there are open sets $U, V \subseteq G$ such that $u \in U$, $v \in V$ and $UV \subseteq W$. We may assume that $U \subseteq A$ and $V \subseteq B$. We can also assume that $U \subseteq \mathbf{d}^{-1}\mathbf{r}(V)$; this simply means that each element of U has a product with some element of V. Then $\mathbf{r}(UV) = \mathbf{r}(U)$ is an open neighbourhood of $\mathbf{r}(uv)$ contained in Z. Since W is a partial bisection, we have that $UV = \mathbf{r}^{-1}\mathbf{r}(UV) \cap W$. But this is an open neighbourhood of uv in UV. It follows that AB is open. □

Lemma 2.4. *Let G be a topological groupoid. If $\mathbf{m}$ is open and G_o is an open subset then $\mathbf{d}$ is a local homeomorphism.*

Proof. We prove first that $\mathbf{d}$ is open. Let $U \subseteq G$ be an open set. We prove that $\mathbf{d}(U)$ is an open subset of G_o. The map $\mathbf{i}$ is a homeomorphism and so U^{-1} is open. Thus $U^{-1} \times U$ is an open set in $G \times G$. Thus $U^{-1} * U = (U^{-1} \times U) \cap (G * G)$ is an open set in $G * G$. It follows that $\mathbf{m}(U^{-1} * U) = U^{-1}U$ is an open set in G and so $U^{-1}U \cap G_o$ is an open set in G_o. Observe that $\mathbf{d}(U) = U^{-1}U \cap G_o$. Thus $\mathbf{d}$ is an open mapping.

We now prove that if $\mathbf{d}$ is open and G_o is open then $\mathbf{d}$ is a local homeomorphism. Let $g \in G$. Then by Lemma 2.1, there is an open partial bisection U such that $g \in U$. The map $\mathbf{d}$ restricted to U gives rise to a bijection $(\mathbf{d}|U)\colon U \to \mathbf{d}(U)$. But $\mathbf{d}$ is an open map and so $(\mathbf{d}|U)$ is a homeomorphism. It follows that $\mathbf{d}$ is a local homeomorphism. □

The obvious question is why should étale groupoids be regarded as a nice class of topological groupoids? The following theorem gives us one reason. We need a little notation: if X is a topological space denote its lattice of open sets by $\Omega(X)$.

Theorem 2.5 (Resende [7]). *Let G be a topological groupoid. Then G is étale if and only if $\Omega(G)$ is a monoid under subset multiplication.*

Proof. Suppose that G is étale. Then by Lemma 2.3, the map $\mathbf{m}$ is open and so $\Omega(G)$ is closed under subset multiplication. By Lemma 2.2, the set G_o is open and this is the identity for subset multiplication. It follows that $\Omega(G)$ is a monoid. To prove the converse, assume that $\Omega(G)$ is a monoid under subset multiplication. This implies that $\mathbf{m}$ is an open map and that G_o is open and so by Lemma 2.4, it follows that $\mathbf{d}$ is a local homeomorphism and so G is étale. □

We may paraphrase the above theorem by saying that étale groupoids are those topological groupoids that have an algebraic alter ego.

Recall that a Boolean space is a 0-dimensional, compact Hausdorff space. We say that an étale topological groupoid is *Boolean* if its space of identities is a Boolean space. Thus Boolean groupoids generalize Boolean spaces since a Boolean space is a Boolean groupoid consisting entirely if identities. We shall regard Boolean groupoids as being *non-commutative Boolean spaces*.

If we combine Theorem 2.5 and Proposition 1.8, we have the following.

Theorem 2.6. *Let G be an étale topological groupoid. Then the set of all open partial bisections forms an inverse monoid.*

There are now two, related paths that one may now take. Our path will be that of inverse semigroups and, in particular, inverse semigroups of open partial bisections. The second path is to take the monoid $\Omega(G)$ as the fundamental structure. This leads to a class of quantales called *inverse quantal frames*. These are the subject of [8]. Intuitively, $\Omega(G)$ can be regarded as the quantale of (abstract) binary relations associated with the inverse monoid of open partial bisections.

References

[1] Ch. Ehresmann, *Oeuvres complètes et commentées*, (ed A. C. Ehresmann) Supplements to *Cahiers de topologie et géométrie différentielle*, Amiens, 1980–83.

[2] R. Exel, Inverse semigroups and combinatorial C^*-algebras, *Bull. Braz. Math. Soc.* (N.S.) **39** (2008), 191–313.

[3] Ph. J. Higgins, *Categories and groupoids*, freely available at `http://www.tac.mta.ca/tac/reprints/articles/7/tr7.pdf`.

[4] S. Mac Lane, *Categories for the working mathematician*, Second Edition, 1998.

[5] A. L. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progress in Mathematics, **170**, Birkhäuser, Boston, 1998.

[6] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer, 1980.

[7] P. Resende, Lectures on étale groupoids, inverse semigroups and quantales, Lecture Notes, August, 2006.

[8] P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

Mark V. Lawson, Department of Mathematics and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UNITED KINGDOM

Email address: `m.v.lawson@hw.ac.uk`