Stochastic PDEs and their numerical approximation

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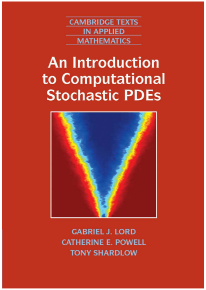
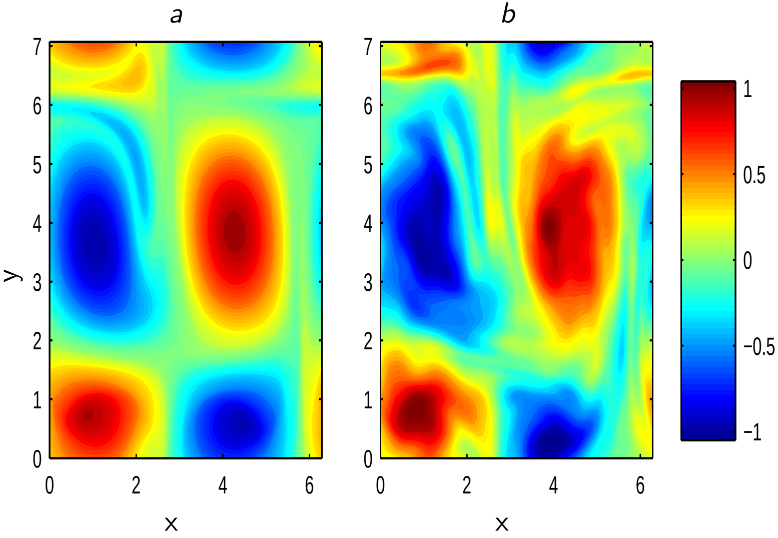
- *Based on*

Introduction to Computational Stochastic Partial Differential Equations
G. J. Lord, C. E. Powell, T. Shardlow
CUP.

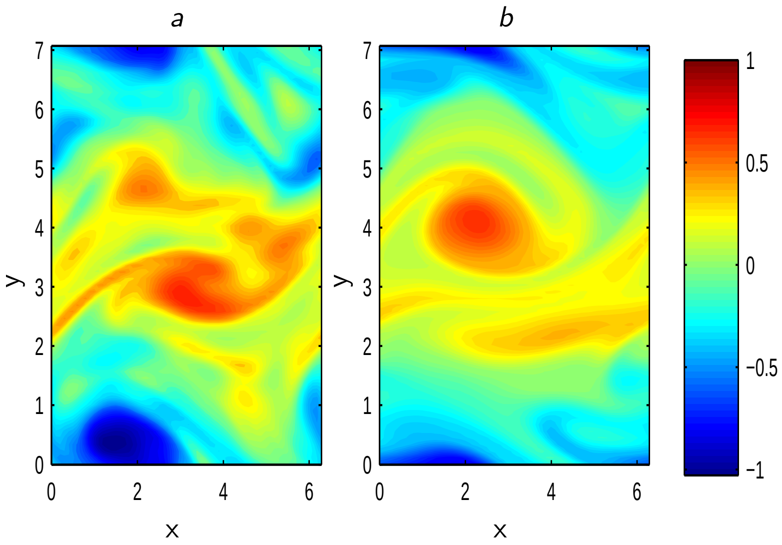


Informal description of SPDEs and numerical approximation.



(a) Deterministic vorticity (b) Stochastic

Informal description of SPDEs and numerical approximation.



(a) Stochastic (Rough)

(b) Stochastic (Smooth)

Informal : will cut some corners !

Some (other) reference books for SPDEs

- Semigroup approach to SPDEs
  - Classic reference :
    Da Prato, Giuseppe and Zabczyk, Jerzy
    Stochastic Equations in Infinite Dimensions
    Encyclopedia of Mathematics and its Applications
    CUP, 1992. ISBN : 0-521-38529-6
  - Chow, Pao-Liu
    Stochastic Partial Differential Equations
    Chapman & Hall/CRC, Boca Raton, FL
    2007, ISBN 978-1-58488-443-9; 1-58488-443-6
- Variational approach
  - Prévôt, Claudia and Röckner, Michael
    A concise course on stochastic partial differential equations
    Springer,2007, ISBN = 978-3-540-70780-6; 3-540-70780-8.

- Numerical methods
- Jentzen, Arnulf and Kloeden, Peter E.
Taylor approximations for stochastic partial differential equations
CBMS-NSF Regional Conference Series in Applied Mathematics
SIAM, 2011, ISBN : 978-1-611972-00-9
- Physics approaches
- García-Ojalvo, Jordi and Sancho, José M.
Noise in spatially extended systems
Springer, ISBN 0-387-98855-6
- C. Gardiner
Stochastic Methods: A handbook for the natural and social sciences
Springer Series in Synergetics
2009, ISBN 978-3-540-70712-7
- SDEs : plenty of choice.
- Øksendal, Bernt,Stochastic Differential Equations,2003.
3-540-04758-1

Background

- PDEs
- ODEs
- SDEs

PDE

Many physical/biological models are described by parabolic PDEs

$$u_t = [\Delta u + f(u)] \quad u(0) = u^0 \text{given} \quad u \in D \tag{1}$$

$+$ BCs on D bounded specified. $f(u)$ given where $u(t, \mathbf{x})$

Two typical examples:

- Nagumo equation

$$u_t = [u_{xx} + u(1-u)(u-\alpha)] \quad u(x,t) \in \mathbb{R},\ x \in [0, L],\ t > 0$$

- Allen-Cahn equation

$$u_t = \left[u_{xx} + u - u^3\right] \quad u(x,t) \in \mathbb{R},\ x \in [0, 2\pi),\ t > 0$$

- We write semilinear PDEs of form

$$u_t = \Delta u + f(u)$$

as ODE on Hilbert space H (eg $L^2(D)$).

$$\frac{du}{dt} = -Au + f(u)$$

$A = -\Delta$

$$u_t = -Au + f(u)$$

Note - we could write solution in three ways :

▶Integrate :

$$u(t) = u(0) + \int_0^t (-Au + f(u))\, ds$$

Too restrictive on regularity of $u(t)$.

▶Weak solution (multiply by test fn. Integ. by parts).

$$\left\langle \frac{du(s)}{dt}, v \right\rangle = -a(u(s), v) + \langle f(u(s)), v \rangle, \qquad \forall v \in V,$$

where $a(u, v) := \langle A^{1/2}u, A^{1/2}v \rangle$

▶Variation of constants

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A} f(u(s)) ds$$

need to understand semigroup e^{-tA} and its properties.

PDE as infinite system of ODEs

$$u_t = -Au + f(u), \quad u(0) = u^0$$

- Look at weak solution

$$\left\langle \frac{du(s)}{dt}, v \right\rangle = -a(u(s), v) + \langle f(u(s)), v \rangle, \qquad \forall v \in V,$$

- Write u as a infinte series

$$u(x,t) = \sum_{k \in \mathbb{Z}} u_k \phi_k(x)$$

with ϕ_k e.func. and λ_k e.val of A (on D +BCs)

- Subst. into PDE, take inner-product with ϕ_k

$$\frac{du_k}{dt} = -\lambda_k u_k + f_k(u), \quad k \in \mathbb{Z}, \quad f(u) = \sum_k f_k(u) \phi_k.$$

Get infinite system of ODEs.
(truncation leads to spectral Galerkin approximation).

- Let's look at adding noise to ODE

ODEs → SDEs & Brownian Motion

In each Fourier mode have ODE of the form :Let's add noise

$$\frac{du}{dt} = \lambda u + f(u) + g(u)\frac{d\beta}{dt}$$

with $\beta_k(t)$ Brownian motion.

$$\beta = (\beta_1(t), \beta_2(t), \cdots, \beta_n(t)), \quad t \geq 0$$

Is a (standard) *Brownian motion* or a *Wiener process* if for each β_j

- $\beta(0) = 0$ a.s.
- Increments $\beta(t) - \beta(s)$ are normal $N(0, t - s)$, for $0 \leq s \leq t$. Equivalently $\beta(t) - \beta(s) \sim \sqrt{t - s}N(0, 1)$.
- Increments $\beta(t) - \beta(s)$ and $\beta(\tau) - \beta(\sigma)$ are independent $0 \leq s \leq t \leq \sigma \leq \tau$.

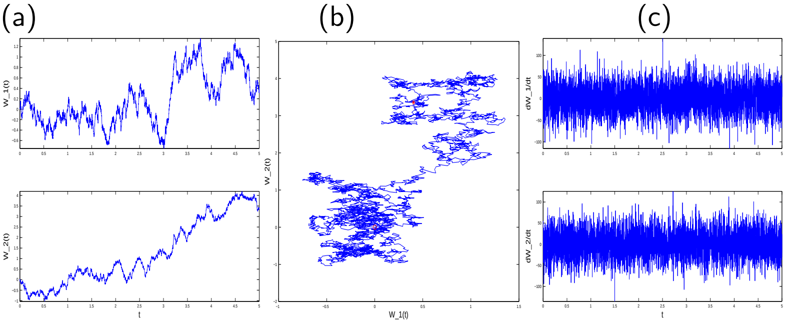
Note : $\beta(t) = \beta(t) - 0 = \beta(t) - \beta(0) \sim N(0, t)$.
So $\mathbf{E}\big[\beta(t)\big] = 0$ and variance $var(\beta(t)) = \mathbf{E}\big[\beta(t)^2\big] = t$.

Actually want a $W(t)$ on a filtered probability space and consider $\mathcal{F}_t$-Brownian motion.

- **probability space** $(\Omega, \mathcal{F}, P)$ consists of a sample space Ω, a set of events $\mathcal{F}$ and a probability measure P.
- **filtered probability space** consists of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $\mathcal{F}_t$ is a filtration of $\mathcal{F}$.
- The **filtration** $\mathcal{F}_t$ is a way of denoting the events that are observable by time t and so as t increases $\mathcal{F}_t$ contains more and more events.
- If $X(t)$, $t \in [0, T]$ is $\mathcal{F}_t$ **adapted** then $X(t)$ is $\mathcal{F}_t$ measurable for all $t \in [0, T]$ (roughly $X(t)$ does not see into the future).
- Finally $X(t)$ is **predictable** if it is $\mathcal{F}_t$ adapted and can be approximated by a sequence $X(s_j) \to X(s)$ if $s_j \to s$ for all $s \in [0, T]$, $s_j < s$.

- Letting $\beta_n \approx \beta(t_n)$, $\Delta\beta_n \sim \sqrt{\Delta t}N(0,1)$

$$\beta_{n+1} = \beta_n + \Delta\beta_n, \qquad n = 1, 2, \ldots, N$$



(a) Two discretised Brownian motions $W_1(t)$, $W_2(t)$ constructed over $[0,5]$ with $N = 5000$ so $\Delta t = 0.001$.
(b) Brownian motion $W_1(t)$ plotted against $W_2(t)$. The paths start at $(0,0)$ and final point at $t = 5$ is marked with a $\star$.
(c) Numerical derivatives of $W_1(t)$ and $W_2(t)$ from (a).

- Path $\beta(t)$ is continuous but not differentiable.

Since $\beta(t)$ is continuous but not differentiable. Understand

$$\frac{du}{dt} = \lambda u + f(u) + g(u)\frac{d\beta}{dt}$$

as integral

$$u(t) = u(0) + \int_0^t (\lambda u(s) + f(u(s)))ds + \int_0^t g(u)d\beta(s).$$

Write as

$$du = [\lambda u + f(u)]\, dt + g(u)dW.$$

Ito stochastic integral $I(t) = \int_0^t g(u)d\beta(s)$

$$I(t) \quad " = " \quad \lim_{N\to\infty} \sum_{n=1}^{N} g(t_{n-1})\Delta\beta_n n$$

The " = " is convergence in mean sqaure

$$\mathbf{E}\left[\|X_j - X\|^2\right] \to 0, \quad as \quad j \to \infty.$$

- Look at Itô integrals only.

The Ito integral satisfies a number of nice properties.

- Martingale property that

$$\mathbf{E}\Big[\int_0^t g(s)d\beta(s)\Big] = 0.$$

- Ito isometry, given in one-dimension by

$$\mathbf{E}\Big[\Big(\int_0^t g(s)d\beta(s)\Big)^2\Big] = \int_0^t \mathbf{E}\big[g(s)^2\big]ds.$$

- But Calculus is different. Chain rule :

Suppose $\frac{du}{dt} = \lambda$. Let $\phi(u) = \frac{1}{2}u^2$. Then

$$\frac{d\phi(u)}{dt} = \frac{d\phi}{du}\,\frac{du}{dt} = u\,\frac{du}{dt} = \lambda u(t).$$

- If $u(t)$ satisfies $du = \lambda\,dt + \sigma\,d\beta(t)$.

The Itô formula says that for $\phi(u) = \frac{1}{2}u^2$.

$$d\phi(u) = u\,du + \frac{\sigma^2}{2}\,dt \tag{2}$$

and we pick up an unexpected extra term $\sigma^2/2dt$.

Itô Formula

Itô SDE : $du = [\lambda u + f(u)]\, dt + g(u) d\beta$

▶Itô formula. $\phi(t,u)$ smooth

$$d\Phi = \frac{\partial \Phi}{\partial t}\, dt + \frac{\partial \Phi}{\partial u}\, du + \frac{1}{2}\frac{\partial^2 \Phi}{\partial u^2} g^2\, dt$$

or written in full

$$\Phi(t,u(t)) = \Phi(0,u_0)$$
$$+ \int_0^t \frac{\partial \Phi}{\partial t}(s,u(s)) + \frac{\partial \Phi}{\partial u}(s,u(s)) f(u(s)) + \frac{1}{2}\frac{\partial^2 \Phi}{\partial u^2}(s,u(s)) g(u(s))^2\, ds$$
$$+ \int_0^t \frac{\partial \Phi}{\partial u}(s,u(s)) g(u(s))\, d\beta(s).$$

▶ Two standard applications : linear equations

▶Ornstein Uhlenbeck (OU) process and

▶Geometric Brownian Motion (GBM)

Example: OU process

$$du = \lambda(\mu - u)dt + \sigma d\beta(t), \qquad u(0) = u_0,$$

for $\lambda, \mu, \sigma \in \mathbb{R}$.
Itô formula with $\Phi(t,u) = e^{\lambda t}\, u$.

$$d\Phi(t,u) = \lambda e^{\lambda t} u dt + e^{\lambda t} du + 0$$

and using the SDE

$$d\Phi(t,u) = \lambda\, e^{\lambda t} u\, dt + e^{\lambda t}\big(\lambda(\mu - u)dt + \sigma d\beta(t)\big).$$

As an integral equation

$$\Phi(t, u(t)) - \Phi(0, u_0) = e^{\lambda t} u(t) - u_0 = \lambda\mu \int_0^t e^{\lambda s}\, ds + \sigma \int_0^t e^{\lambda s}\, d\beta(s).$$

After evaluating the deterministic integral, we find

$$u(t) = e^{-\lambda t} u_0 + \mu\big(1 - e^{-\lambda t}\big) + \sigma \int_0^t e^{\lambda(s-t)} d\beta(s)$$

and this is known as the **variation of constants** solution.

$u(t) = e^{-\lambda t}u_0 + \mu\big(1 - e^{-\lambda t}\big) + \sigma \int_0^t e^{\lambda(s-t)}\, d\beta(s)$
Using the mean zero property of the Itô integral

$$\mu(t) = \mathbf{E}\big[u(t)\big] = e^{-\lambda t}u(0) + \mu(1 - e^{-\lambda t})$$

so that $\mu(t) \to \mu$ as $t \to \infty$ and the process is "mean reverting". For the covariance, first note that

$$\begin{aligned}
\operatorname{Cov} u(t), u(s) =& \mathbf{E}\big[\big(u(s) - \mathbf{E}\big[u(s)\big]\big)\,\big(u(t) - \mathbf{E}\big[u(t)\big]\big)\big] \\
=& \mathbf{E}\big[\int_0^s \sigma e^{\lambda(r-s)} d\beta(r) \int_0^t \sigma e^{\lambda(r-t)} d\beta(r)\big] \\
=& \sigma^2 e^{-\lambda(s+t)} \mathbf{E}\big[\int_0^s e^{\lambda r} d\beta(r) \int_0^t e^{\lambda r} d\beta(r)\big].
\end{aligned}$$

Then, can show using the Itô isometry

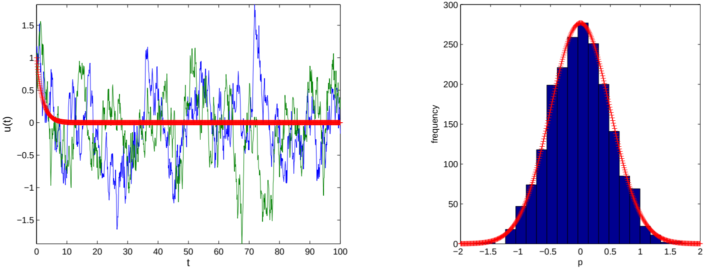
$$\operatorname{Cov} u(t), u(s) = \frac{\sigma^2}{2\lambda} e^{-\lambda(s+t)}\big(e^{2\lambda \min(s,t)} - 1\big).$$

In particular, the variance

$$\operatorname{Var} u(t) = \sigma^2(1 - e^{-2\lambda t})/2\lambda.$$

Then, $\operatorname{Var} u(t) \to \sigma^2/2\lambda$ and $u(t) \to N(\mu, \sigma^2/2\lambda)$ in distribution as $t \to \infty$.

Example: OU process



(a) (b)

(a) Two numerical solutions of the OU SDE and ODE
$u(0) = 1$, $\lambda = 0.5$ and $\sigma = 0.5$.
In (b) we examine the distribution at $t = 100$ showing a histogram from 2000 different realisations.

- Will OU use later for stochastic heat equation.

Example: Geometric Brownian Motion

$$du = r\,u\,dt + \sigma\,u\,d\beta(t),$$

Solution :

$$u(t) = \exp\Big((r - \sigma^2/2)t + \sigma\beta(t)\Big)u_0.$$

By the Itô formula with $\Phi(t,u) = \phi(u) = \log(u)$,

$$d\phi(u) = r\,dt + \sigma\,d\beta(t) - \frac{1}{2}\sigma^2\,dt.$$

Hence,

$$\phi(u(t)) = \phi(u_0) + \int_0^t \Big(r - \frac{\sigma^2}{2}\Big)\,ds + \int_0^t \sigma\;d\beta(s)$$

and $\log u(t) = \log(u_0) + (r - \frac{1}{2}\sigma^2)t + \sigma\beta(t)$.
Taking the exponential, get result.

Systems of SDEs : $\mathbf{u} \in \mathbb{R}^d$.

- Given drift $\mathbf{f}(\mathbf{u}) : \mathbb{R}^d \to \mathbb{R}^d$
- Diffusion $G(\mathbf{u}) : \mathbb{R}^d \to \mathbb{R}^{d \times m}$
- $\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(2), \ldots, \beta_m(t))^T \in \mathbb{R}^m$.

We write SDE as

$$d\mathbf{u} = \mathbf{f}(\mathbf{u})dt + G(\mathbf{u})d\boldsymbol{\beta}(t)$$

for integral

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{f}(\mathbf{u}(s))ds + \int_0^t G(\mathbf{u}(s))d\boldsymbol{\beta}(s).$$

Approximate the Ito Stochastic DE:

SDE is an integral equation:

$$u(t) = u(0) + \int_0^t \left[\lambda u + f(u(s))\right] ds + \int_0^t g(u(s)) d\beta(s).$$

- Let's get a numerical scheme : 1 step $t = \Delta t$

$$u(t) = u(0) + \int_0^t (\lambda u + f(u(s))) ds + \int_0^t g(u(s)) d\beta(s).$$

$$u(\Delta t) = u(0) + \int_0^{\Delta t} \left[\lambda u(s) + f(u(s))\right] ds + \int_0^{\Delta t} g(u(s)) d\beta(s).$$

$$u(\Delta t) \approx u(0) + \left[\lambda u(0) + f(u(0))\right] \int_0^{\Delta t} ds + g(u(0)) \int_0^{\Delta t} d\beta(s).$$

$$u(\Delta t) \approx u(0) + \Delta t \left[\lambda u(0) + f(u(0))\right] + g(u(0)) \Delta\beta_1.$$

$$u(\Delta t) \approx u(0) + \Delta t \left[\lambda u(0) + f(u(0))\right] + \sqrt{\Delta t} g(u(0)) \xi.$$

where $\xi \sim N(0, 1)$.

$$u^{n+1} = u^n + \Delta t \left[\lambda u^n + f(u^n)\right] + \sqrt{\Delta t} g(u^n) \xi$$

Stability: GBM $du = ru\,dt + \sigma u\;d\beta$

From solution of GBM see that $\mathbf{E}\big[u(t)^2\big] = e^{(2r+\sigma^2)t}u_0^2$.
Thus:

$$\mathbf{E}\big[u(t)^2\big] \to 0 \text{ provided } r + \sigma^2/2 < 0.$$

EM method : $u_{n+1} = u_n + ru_n\,\Delta t + \sigma u_n\,\Delta\beta_n$.

$$u_n = \prod_{j=0}^{n-1}\Big(1 + r\Delta t + \sigma\Delta\beta_j\Big)\,u_0.$$

Second moment of u_n is (using $\Delta\beta_j$ *iid*)

$$\mathbf{E}\big[u_n^2\big] = \prod_{j=0}^{n-1}\Big((1 + r\Delta t)^2 + \sigma^2\Delta t)\Big)u_0^2,$$

Thus $\mathbf{E}\big[u_n^2\big] \to 0$ as $n \to \infty$ if and only if

$$\Big|\big(1 + r\,\Delta t\big)^2 + \sigma^2\Delta t\Big| = 1 + 2\Delta t\big(r + \sigma^2/2 + \Delta t\,r^2/2\big) < 1.$$

ie get a restriction on step size : $0 < \Delta t < -2\big(r + \sigma^2/2\big)/r^2$.

Convergence: Strong & Weak

- *Strong convergence* :

$$\sup_{0 \le t_n \le T} \|\mathbf{u}(t_n) - \mathbf{u}_n\|_{L^2(\Omega,\mathbb{R}^d)} = \sup_{0 \le t_n \le T} \left(\mathbf{E}\|\mathbf{u}(t_n) - \mathbf{u}_n\|_2^2\right)^{1/2} \le \Delta t^p.$$

Care about approximating the sample path $\mathbf{u}(\cdot, \omega)$ Euler Maruyama
:
$O(\Delta t^{1/2})$ multiplicative noise
$O(\Delta t^1)$ additive noise.

- *Weak convergence* : Estimate $\mathbf{E}\left[\phi(\mathbf{u}(T))\right]$

$$\mu_M := \frac{1}{M} \sum_{j=1}^{M} \phi(\mathbf{u}_N^j).$$

$$\mathbf{E}\left[\phi(\mathbf{u}(T))\right] - \mu_M = \underbrace{\left[\mathbf{E}\left[\phi(\mathbf{u}(T))\right] - \mathbf{E}\left[\phi(\mathbf{u}_N)\right]\right]}_{\text{weak discretization error}} + \underbrace{\left[\mathbf{E}\left[\phi(\mathbf{u}_N)\right] - \mu_M\right]}_{\text{Monte Carlo error}}.$$

Care about the distributions. EM weak error $O(\Delta t)$.

Recap

- PDE - $u_t = [\Delta u + f(u)]$
  - Solutions : Weak solution & Variations of Constants
  - PDE as infinte system of ODEs
- SDEs : $du = [\lambda u + f(u)]\,dt + g(u)dW$
  - Brownian motion & Ito integrals
  - OU and GBM SDEs
  - EM approximation

$$v^{n+1} = v^n + \Delta t(\lambda v^n + f(v^n)) + \sqrt{\Delta t}g(v^n)\xi, \qquad \xi \sim N(0,1).$$

  - Stability : may need (semi-)implcit method.
  - Convergence

$$\text{SPDE } u_t = [\Delta u + f(u)] + g(u)W_t$$

- Introduce noise and covariance Q
- Introduce stochastic integral
- Solution
- Discretization

Some example SPDEs

- What is an SPDE ?
PDEs with forcing that is random in both space and time.

- They include random fluctuations that occur in nature and are missing in deterministic PDE descriptions.

- Example :
Heat equation with a random term $\zeta(t, \mathbf{x})$

$$u_t = \Delta u + \zeta(t, \mathbf{x}), \qquad t > 0, \quad \mathbf{x} \in D,$$

We will choose $\zeta = W_t$, where $W(t, \mathbf{x}, \omega)$ is a Wiener process.
Write SPDE as

$$du = \Delta u dt + dW, \qquad t > 0, \quad \mathbf{x} \in D,$$

PDE + Additive Noise

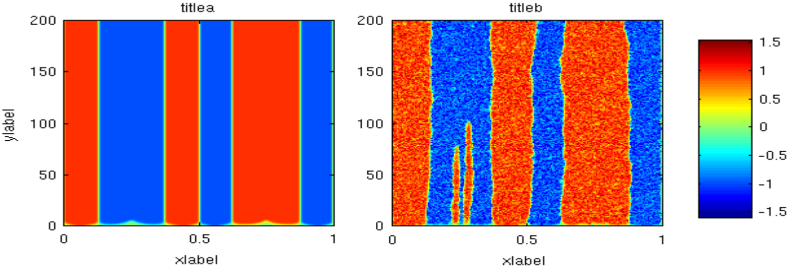
Want to examine effects of noise $W(x,t)$

$$du = [\Delta u + f(u)]\,dt + g(u)dW$$

- In time dW is white (formally derivative of Brownian motion).
- In space either white or colored.
- Additive (or external) noise : $g(u) = \nu$ constant

eg Allen–Cahn & random external fluctuations :

$$du = \left[u_{xx} + u - u^3\right] dt + \nu dW$$



PDE + Multiplicative Noise

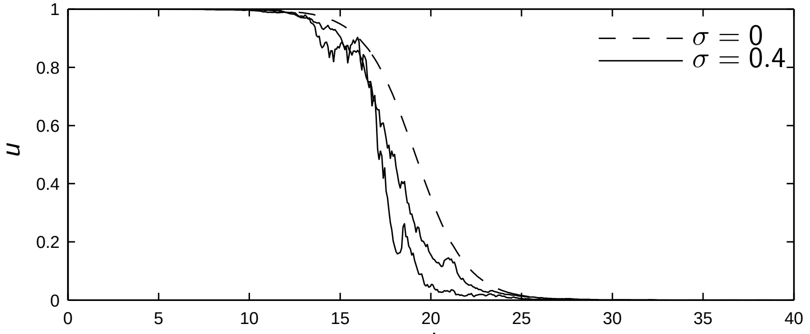
- Multiplicative (or intrinsic) noise $g(u)$

eg Nagumo & noise on parmeter α

$$u_t = [u_{xx} + u(1-u)(u-\alpha)]$$

$$u_t = [u_{xx} + u(1-u)(u-\alpha+\sigma W_t)]$$

$$du = [u_{xx} + u(1-u)(u-\alpha)]\,dt + \sigma u(u-1)dW$$



Vorticity

- model for large scale flows, e.g. related to climate modelling or the evolution of the red spot on Jupiter.

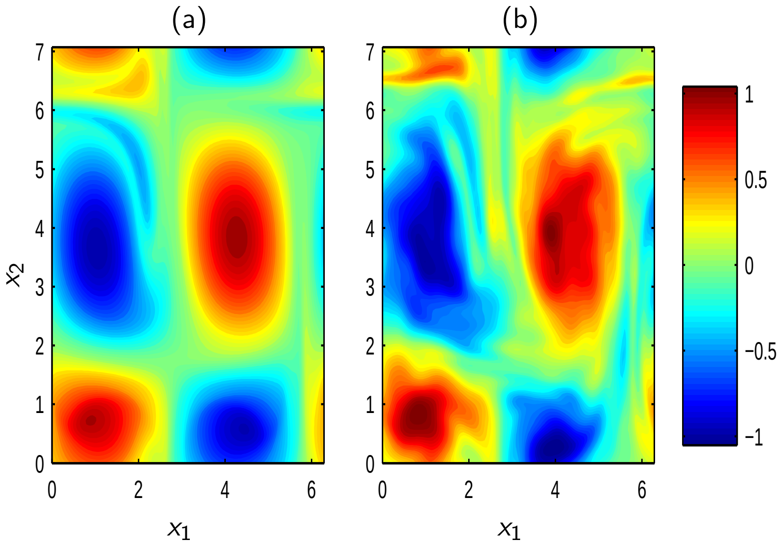
In two dimensions, the vorticity $u := \nabla \times \mathbf{v}$ satisfies the PDE

$$u_t = \varepsilon \, \Delta u - (\mathbf{v} \cdot \nabla) u \tag{3}$$

where $\Delta \psi = -u$, $\psi(t, \mathbf{x})$ is the scalar stream function, and $\mathbf{v} = (\psi_y, -\psi_x)$.

- Additive noise captures small scale perturbations.

$$du = \left[\varepsilon \, \Delta u - (\mathbf{v} \cdot \nabla) u \right] dt + \sigma \, dW(t). \tag{4}$$



Deterministic

Stochastic

filtering and sampling

- Suppose we have a signal $Y(x)$, $x \geq 0$,

$$dY = f(Y(x))\,dx + \sqrt{\sigma}\,d\beta_1(x), \qquad Y(0) = 0, \tag{5}$$

$f : \mathbb{R} \to \mathbb{R}$ is a given forcing term,
$\beta_1(x)$ is a Brownian motion,
σ controls the strength of the noise.

- Noisy observations $Z(x)$ of the signal $Y(x)$.

$$dZ = Y(x)\,dx + \sqrt{\gamma}\,d\beta_2(x), \qquad Z(0) = 0, \tag{6}$$

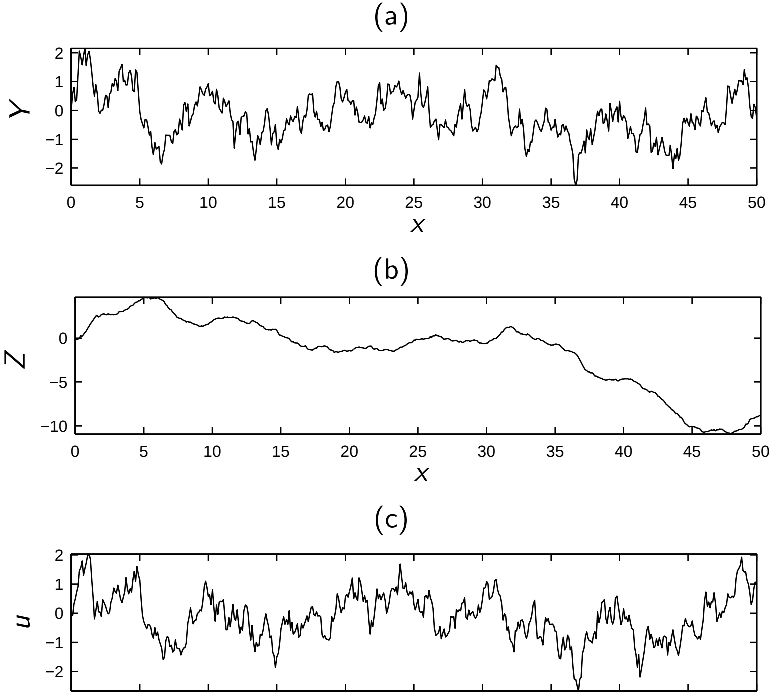
$\beta_2(x)$ is also a Brownian motion (independent of β_1)
γ determines the strength of the noise in the observation.
If $\gamma = 0$, we observe the signal exactly.

- Goal :
Estimate the signal $Y(x)$ given observations $Z(x)$ for $x \in [0, b]$.

Can get estimate of signal from long time simulation of

$$du = \left[\frac{1}{\sigma}\left(u_{xx} - f(u)f'(u) - \frac{\sigma}{2}f''(u)\right)\right]dt + \frac{1}{\gamma}\left[\frac{dY}{dx} - u\right]dt + \sqrt{2}\,dW(t) \tag{7}$$

for $(t, x) \in (0, \infty) \times [0, b]$ and where $W(t)$ is a space-time Wiener process.
Since $Y(x)$ is only Hölder continuous with exponent less than $1/2$, the derivative $\frac{dY}{dx}$ and the SPDE (7) require careful interpretation.


(a)
Y
x
(b)
Z
x
(c)
u

- We now introduce for SPDEs
  - the noise $W(t, \mathbf{x}, \omega) = W(t, \mathbf{x}) = W(t)$
  - stochastic Itô integral

Wiener process

- Want to introduce space dependence into Brownian motion.

Instead of working in $L^2(D)$ we develop theory on separable Hilbert space U (so has orthonormal basis).
Denote norm $\|\cdot\|_U$ and inner product $\langle \cdot, \cdot \rangle_U$

- We start by defining $W(t, \mathbf{x})$ where W has some spatial correlation.

We define the space $L^2(\Omega, H)$:

$$\|X\|^2_{L^2(\Omega,H)} = \mathbf{E}\left[(\|X\|_H)^2\right] < \infty.$$

Q-Wiener process

▶ Q-Wiener process $\{W(t)\colon t \geq 0\}$ is a U-valued process.
Each $W(t)$ is a U-valued Gaussian random variable and each has a well-defined covariance operator.
The covariance operator at $t = 1$ is denoted Q.

Assumption

$Q \in \mathcal{L}(U)$ is

- *non-negative ($\langle u, Qu \rangle \geq 0$)*
- *symmetric ($\langle u, Qu \rangle = \langle Qu, u \rangle$)*
- *Q has orthonormal basis $\{\chi_j\}_{j \in \mathbb{N}}$ of eigenfunctions. Corresponding eigenvalues $q_j \geq 0$. $Q\chi_j = q_j \chi_j$.*
- *Q is trace class i.e.*

$$\sum_{j=1}^{\infty} q_j < \infty.$$

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space.

- The **filtration** $\mathcal{F}_t$ is a way of denoting the events that are observable by time t and so as t increases $\mathcal{F}_t$ contains more and more events.
- If $X(t)$, $t \in [0, T]$ is $\mathcal{F}_t$ **adapted** then $X(t)$ is $\mathcal{F}_t$ measurable for all $t \in [0, T]$ (roughly $X(t)$ does not see into the future).

Definition (Q-Wiener process)

Let Q satisfy the Assumption. A U-valued stochastic process $\{W(t) \colon t \geq 0\}$ is a *Q-Wiener process* if

1. $W(0) = 0$ a.s.,
2. $W(t)$ is a continuous function $\mathbb{R}^+ \to U$, for each $\omega \in \Omega$.
3. $W(t)$ is $\mathcal{F}_t$-adapted and $W(t) - W(s)$ is independent of $\mathcal{F}_s$, $s < t$,
4. $W(t) - W(s) \sim N(0, (t-s)Q)$ for all $0 \leq s \leq t$.

Q-Wiener expansion

We now characterise a Q-Wiener process in a useful way.

Theorem

Let Q satisfy the Assumption on noise.
Then $W(t)$ is a Q-Wiener process if and only if

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \chi_j\, \beta_j(t), \qquad a.s., \tag{8}$$

where $\beta_j(t)$ are iid $\mathcal{F}_t$-Brownian motions.
The series converges in $L^2(\Omega, U)$.

Proof : 1) Let $W(t)$ be a Q-Wiener process.
Since $\{\chi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for U,

$$W(t) = \sum_{j=1}^{\infty} \langle W(t), \chi_j \rangle_U\, \chi_j.$$

Let $\beta_j(t) := \frac{1}{\sqrt{q_j}} \langle W(t), \chi_j \rangle_U$, so that (8) holds.

Sketch of proof

2) Let's show $W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \chi_j\, \beta_j(t)$, converges in $L^2(\Omega, U)$. Consider the finite sum approximation

$$W^J(t) := \sum_{j=1}^{J} \sqrt{q_j}\, \chi_j\, \beta_j(t) \tag{9}$$

- By orthonormality of eigenfunctions χ_j & Parseval's identity

$$\left\| W^J(t) - W^M(t) \right\|_U^2 = \sum_{j=M+1}^{J} q_j\, \beta_j(t)^2. \tag{10}$$

- Each $\beta_j(t)$ is a Brownian motion. Taking the expectation gives

$$\mathbf{E}\left[\left\| W^J(t) - W^M(t) \right\|_U^2 \right] = \sum_{j=M+1}^{J} q_j\, \mathbf{E}\left[\beta_j(t)^2 \right] = t \sum_{j=M+1}^{J} q_j.$$

As Q is trace class, $\sum_{j=1}^{\infty} q_j < \infty$, and RHS $\to 0$ as $M, J \to \infty$.

Example $W(t, x)$

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \chi_j\, \beta_j(t), \qquad a.s.,$$

Let's take $U = L^2(D)$ for some domain D. Eg $D = (0, 1)$.
We have

$$W(t, x) = \sum_{j=1}^{\infty} \sqrt{q_j}\, \chi_j(x)\, \beta_j(t),$$

We can specify eigenfunctions $\chi_j(x)$ and eigenvalues q_j with appropriate decay rate.

- Let's construct $W(t) \in H_0^r(0, 1)$.

Take $\chi_j(x) = \sqrt{2}\sin(j\pi x)$ and $q_j = |j|^{-(2r+1+\epsilon)}$ for some $\epsilon > 0$.
So get

$$W(t, x) = \sum |j|^{-(2r+1+\epsilon)/2} \sqrt{2}\sin(j\pi x)\beta_j(t).$$

$$W(t,x) = \sum |j|^{-(2r+1+\epsilon)/2}\sqrt{2}\sin(j\pi x)\beta_j(t).$$

$W(t) \in H_0^r(0,1)$.

Check: For $r = 0$: $W(t) \in L^2(\Omega, L^2(D))$

$$\|W\|_{L^2(D)}^2 = \sum |j|^{-(1+\epsilon)}\beta_j(t)^2$$

$$\mathbf{E}\left[\|W\|_{L^2(D)}^2\right] = \sum t|j|^{-(1+\epsilon)}.$$

Check: For $r = 1$: $W(t) \in L^2(\Omega, H_0^1(D))$

$$W_x(t) = \sum |j|^{-(2+1+\epsilon)/2} j\pi\sqrt{2}\cos(j\pi x)\beta_j(t).$$

$$\|W_x\|_{L^2(D)}^2 = C\sum |j|^{-(2+1+\epsilon)} j^2\beta_j(t)^2$$

$$\mathbf{E}\left[\|W_x\|_{L^2(D)}^2\right] = C\sum t|j|^{-(1+\epsilon)}.$$

Approximation of $W(t,x)$

Assume eigenfunctions χ_j and eigenvalues q_j of Q are known.
Use finite sum to approximate $W(t)$:

$$W(t) \approx W^{J-1}(t) := \sum_{j=1}^{J-1} \sqrt{q_j}\, \chi_j\, \beta_j(t).$$

Can compute increments of W by

$$W^{J-1}(t_{n+1}) - W^{J-1}(t_n) = \sqrt{\Delta t_{\text{ref}}} \sum_{j=1}^{J-1} \sqrt{q_j}\, \chi_j\, \zeta_j^n.$$

$\zeta_j^n \sim N(0,1)$.
To compute same sample path with larger time step $\Delta t = \kappa \Delta t_{\text{ref}}$

$$W^J(t+\Delta t) - W^J(t) = \sum_{n=0}^{\kappa-1} \Big(W^J(t+t_{n+1}) - W^J(t+t_n) \Big).$$

Example $W(t) \in H_0^r(0, a)$

$$W(t) \approx W^{J-1}(t) := \sum_{j=1}^{J-1} \sqrt{q_j}\,\sqrt{2}\sin(j\pi x)\,\beta_j(t).$$

For effiecency use Discrete Sine Transform.
Sample $W(t, x)$ at $x_k = ka/J$, $k = 1, 2, \ldots, J - 1$.

```
>> dtref=0.01;  kappa=100; r=1/2; J=128; a=1;
>> bj=get_onedD_bj(dtref,J,a,r);
>> dW=get_onedD_dW(bj,kappa,0,1);
```

```
function bj = get_onedD_bj(dtref,J,a,r)
jj  = [1:J-1]'; myeps=0.001;
root_qj=jj.^-((2*r+1+myeps)/2);% set decay for H^r
bj=root_qj*sqrt(2*dtref/a);
```

Code to form the coefficients b_j.

▶Inputs are `dtref`$= \Delta t_{\mathrm{ref}}$, `J`$= J$, the domain size `a`, and regularity parameter `r`$= r$.

▶Output is a vector `bj` of coefficients b_j, $j = 1, \ldots, J-1$.

Here we fix $\epsilon = 0.01$ in the definition of q_j using `myeps`.

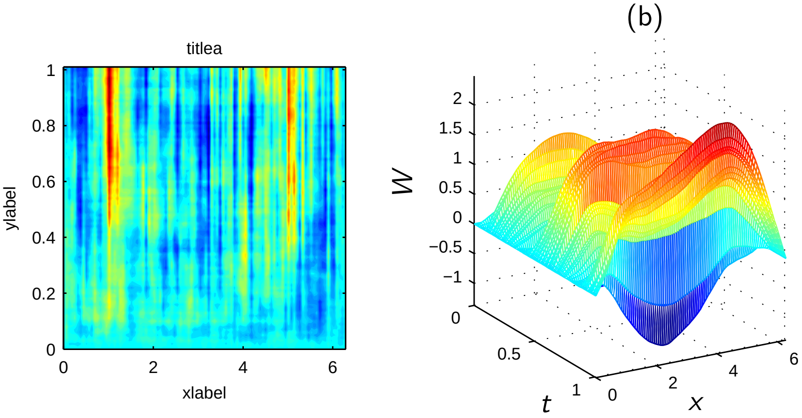
```
function dW=get_onedD_dW(bj,kappa,iFspace,M)
if(kappa==1)
  nn=randn(length(bj),M);
else
  nn=squeeze(sum(randn(length(bj),M,kappa),3));
end
X=bsxfun(@times,bj,nn);
if(iFspace==1)
  dW=X;
else
  dW=dst1(X);
end
```

Code to sample $W^{J-1}(t+\kappa\Delta t_{\mathrm{ref}}, x_k) - W^{J-1}(t, x_k)$

▶Inputs are : coefficients `bj`, `kappa` $= \kappa$, a flag `iFspace`, and the number `M` of independent realisations to compute.

▶If `iFspace=0`, the output `dW` is a matrix of M columns with kth entry $W^J(t+\kappa\Delta t_{\mathrm{ref}}, x_k) - W^J(t, x_k)$ for $k = 1, \ldots, J-1$.

▶If `iFspace=1` then the columns of `dW` are the inverse DST of those for `iFspace=0`.



Approximate sample paths of the Q-Wiener process $W(t) \in H_0^r(0,1)$.

(a) $r = 0$ and (b) $r = 2$.

Generated with $J = 128$ and $\Delta t_{\text{ref}} = 0.01$.

In each case $W(t,0) = W(t,1) = 0$.

Q-Wiener process in two dimensions

Let $D = (0, a_1) \times (0, a_2)$ and $U = L^2(D)$.
Consider $Q \in \mathcal{L}(U)$ with eigenfunctions $\chi_{j_1,j_2}(\mathbf{x}) = \frac{1}{\sqrt{a_1 a_2}} e^{2\pi \mathbf{i} j_1 x_1/a_1} e^{2\pi \mathbf{i} j_2 x_2/a_2}$ and, for a parameter $\alpha > 0$ and $\lambda_{j_1,j_2} = j_1^2 + j_2^2$, eigenvalues

$$q_{j_1,j_2} = e^{-\alpha \lambda_{j_1,j_2}}.$$

For even integers J_1, J_2, let

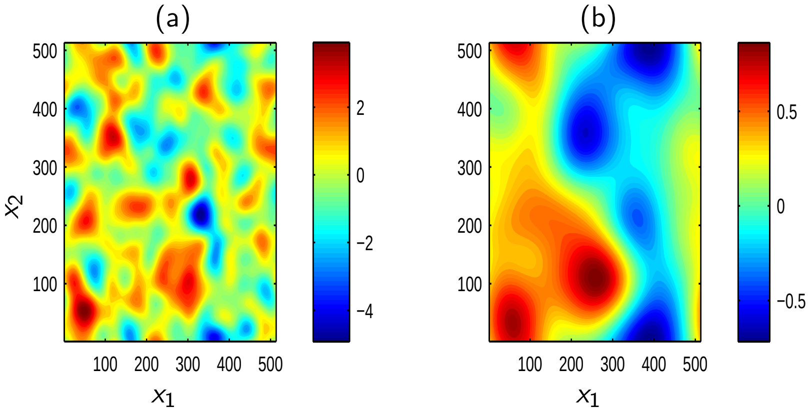
$$W^J(t, \mathbf{x}) := \sum_{j_1=-J_1/2+1}^{J_1/2} \sum_{j_2=-J_2/2+1}^{J_2/2} \sqrt{q_{j_1,j_2}}\, \chi_{j_1,j_2}(\mathbf{x})\, \beta_{j_1,j_2}(t),$$

We generate two independent copies of $W^J(t, \mathbf{x}_{k_1,k_2})$ using a single FFT.

```
>> J=[512,512]; dtref=0.01; kappa=100;  a=[2*pi,2*pi];
>> alpha=0.05; bj = get_twod_bj(dtref,J,a,alpha);
>> [W1,W2]=get_twod_dW(bj,kappa,1);
```

```
function bj=get_twod_bj(dtref,J,a,alpha)
lambdax= 2*pi*[0:J(1)/2 -J(1)/2+1:-1]'/a(1);
lambday= 2*pi*[0:J(2)/2 -J(2)/2+1:-1]'/a(2);
[lambdaxx lambdayy]=meshgrid(lambday,lambdax);
root_qj=exp(-alpha*(lambdaxx.^2+lambdayy.^2)/2);    % smooth
bj=root_qj*sqrt(dtref)*J(1)*J(2)/sqrt(a(1)*a(2));
```

```
function [dW1,dW2]=get_twod_dW(bj,kappa,M)
J=size(bj);
if(kappa==1)
  nnr=randn(J(1),J(2),M);  nnc=randn(J(1),J(2),M);
else
  nnr=squeeze(sum(randn(J(1),J(2),M,kappa),4));
  nnc=squeeze(sum(randn(J(1),J(2),M,kappa),4));
end
nn2=nnr + sqrt(-1)*nnc; TMPHAT=bsxfun(@times,bj,nn2);
tmp=ifft2(TMPHAT); dW1=real(tmp); dW2=imag(tmp);
```



(a) $\alpha = 0.05$ and (b) $\alpha = 0.5$

Computed with $J_1 = J_2 = 512$ and at $t = 1$.

Both processes take values in $H^r((0, 2\pi) \times (0, 2\pi))$ for any $r \geq 0$.

Cylindrical Wiener process

When $Q = I$, $q_j = 1$ for all j then

$$W(t) = \sum_{j=1}^{\infty} \chi_j \, \beta_j(t)$$

This is *white noise* in space.

▶ Analogy with white light : homogeneous mix ($q_j = 1$) of all eigenfunctions.

▶ For a Q-Wiener process is coloured noise and the heterogeneity of the eigenvalues q_j causes correlations in space.

Problem:

However Q is not trace class on U so series does not converge.

▶ Trick :

Introduce U_1 such that $U \subset U_1$ and $Q = I$ is a trace class operator when extended to U_1.

Definition (cylindrical Wiener process)

Let U be a separable Hilbert space.
The *cylindrical Wiener process* (also called *space-time white noise*) is the process $W(t)$ defined by

$$W(t) = \sum_{j=1}^{\infty} \chi_j \, \beta_j(t), \tag{11}$$

where $\{\chi_j\}_{j=1}^{\infty}$ is *any* orthonormal basis of U and $\beta_j(t)$ are *iid* $\mathcal{F}_t$-Brownian motions.

If $U \subset U_1$ for a second Hilbert space U_1, the series converges in $L^2(\Omega, U_1)$ if the inclusion $\iota \colon U \to U_1$ is Hilbert–Schmidt.

Itô integral

We now define for $W(t)$ Q-Wiener process

$$I(t) = \int_0^t B(s)\, dW(s)$$

$W(t)$ takes values in the space U.
Will consider SPDEs in a Hilbert space H so want I to take values in H.
Thus want B that are $\mathcal{L}(U_0, H)$-valued processes, for $U_0 \subset U$.

Definition (L_0^2 space for integrands)

Let $U_0 = \{Q^{1/2}u \colon u \in U\}$ for $Q^{1/2}$.
L_0^2 is the set of linear operators $B \colon U_0 \to H$ such that

$$\|B\|_{L_0^2} := \left(\sum_{j=1}^{\infty} \|BQ^{1/2}\chi_j\|^2 \right)^{1/2} = \|BQ^{1/2}\|_{HS(U,H)} < \infty,$$

where χ_j is an orthonormal basis for U.
L_0^2 is a Banach space with norm $\|\cdot\|_{L_0^2}$.

The truncated form $W^J(t)$ of the Q-Wiener process is finite-dimensional and the integral

$$\int_0^t B(s)\, dW^J(s) = \sum_{j=1}^{J} \int_0^t B(s)\sqrt{q_j}\,\chi_j\, d\beta_j(s) \qquad (12)$$

is well-defined.

- We can show the limit as $J \to \infty$ of (12) exists in $L^2(\Omega, H)$.

Define the stochastic integral by

$$\int_0^t B(s)\, dW(s) := \sum_{j=1}^{\infty} \int_0^t B(s)\sqrt{q_j}\,\chi_j\, d\beta_j(s). \qquad (13)$$

Semilinear SPDEs

$$du = \Big[-Au + f(u)\Big]\,dt + G(u)\,dW(t), \qquad u(0) = u_0 \in H,$$

Global Lipschitz $f : H \to H$, $G : H \to L^2_0$.

Assumption

Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $-A : \mathcal{D}(A) \subset H \to H$.
Suppose that A has a complete orthonormal set of eigenfunctions $\{\phi_j : j \in \mathbb{N}\}$ and eigenvalues $\lambda_j > 0$, ordered so that $\lambda_{j+1} \geq \lambda_j$.

Example: Stochastic Heat Equation with homogeneous Dirichlet BCs. Here $H = U = L^2(0, \pi)$,

$$du = \Delta u dt + \sigma\, dW(t), \quad u(0) = u_0 \in L^2(0, \pi)$$

$A = -\Delta$ with domain $\mathcal{D}(A) = H^2(0, \pi) \cap H^1_0(0, \pi)$.
Eigenvalues of A are $\lambda_j = j^2$. A satisfies the Assumption.
A is the generator of an infinitesimal semigroup $S(t) = e^{-tA}$.
$f = 0$, and $G(u) = \sigma I$, so that $G(u)v = \sigma v$ for $v \in U$ and we have additive noise.

Solution: Strong

$$du = \Big[-Au + f(u)\Big]\,dt + G(u)\,dW(t)$$

Definition (strong solution)

A predictable H-valued process $\big\{u(t)\colon t \in [0, T]\big\}$ is called a *strong solution* if

$$u(t) = u_0 + \int_0^t \Big[-Au(s) + f(u(s))\Big]\,ds + \int_0^t G(u(s))\,dW(s), \quad \forall t \in [0, T].$$

- Too restrictive in practice as need $u(t) \in \mathcal{D}(A)$.

Weak Solution: $du = \Big[-Au + f(u)\Big]\, dt + G(u)\, dW(t)$

Definition (weak solution)

A predictable H-valued process $\big\{u(t)\colon t \in [0,T]\big\}$ is a *weak solution* if

$$\begin{aligned}\big\langle u(t), v\big\rangle =& \big\langle u_0, v\big\rangle + \int_0^t \Big[-\big\langle u(s), Av\big\rangle + \big\langle f(u(s)), v\big\rangle\Big]\, ds \\ &+ \int_0^t \big\langle G(u(s))\, dW(s), v\big\rangle, \quad \forall t \in [0,T],\ v \in \mathcal{D}(A),\end{aligned}$$

where

$$\int_0^t \big\langle G(u(s))\, dW(s), v\big\rangle := \sum_{j=1}^{\infty} \int_0^t \Big\langle G(u(s)) \sqrt{q_j}\, \chi_j, v\Big\rangle\, d\beta_j(s).$$

- ‘weak’ refers to the PDE, not to the probabilistic notion of weak solution.
- (No condition on du/dt, the test space is $\mathcal{D}(A)$, and $u(t) \in H$)

stochastic heat equation (SHE) in one dimension

$$du = \Delta u dt + \sigma\, dW(t), \quad u(0) = u_0 \in L^2(0,\pi)$$
$$du = -Au dt + \sigma\, dW(t), \quad u(0) = u_0 \in L^2(0,\pi)$$

$-A$ has e.funcs $\phi_j(x) = \sqrt{2/\pi}\sin(jx)$ and e.vals $\lambda_j = j^2$ for $j \in \mathbb{N}$.

▶ Suppose for $W(t)$ the eigenfunctions χ_j of Q are same as the eigenfunctions ϕ_j of A.

Weak solution satisfies, $v \in \mathcal{D}(A)$,

$$\left\langle u(t), v \right\rangle_{L^2(0,\pi)} = \left\langle u_0, v \right\rangle_{L^2(0,\pi)} + \int_0^t \left\langle -u(s), Av \right\rangle_{L^2(0,\pi)} ds + \sum_{j=1}^{\infty} \int_0^t \sigma \sqrt{q_j} \left\langle \phi_j, v \right\rangle_{L^2(0,\pi)} d\beta_j(s).$$

Write $u(t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\phi_j$ for $\hat{u}_j(t) := \langle u(t), \phi_j \rangle_{L^2(0,\pi)}$.
Take $v = \phi_j$, to get

$$\hat{u}_j(t) = \hat{u}_j(0) + \int_0^t (-\lambda_j)\hat{u}_j(s)\, ds + \int_0^t \sigma\sqrt{q_j}\, d\beta_j(s).$$

$$\hat{u}_j(t) = \hat{u}_j(0) + \int_0^t (-\lambda_j)\hat{u}_j(s)\,ds + \int_0^t \sigma\sqrt{q_j}\,d\beta_j(s), j \in \mathbb{N}$$

- Hence, $\hat{u}_j(t)$ satisfies the SDE

$$d\hat{u}_j = -\lambda_j \hat{u}_j\,dt + \sigma\sqrt{q_j}\,d\beta_j(t).$$

Each coefficient $\hat{u}_j(t)$ is an Ornstein–Uhlenbeck (OU) process which is a Gaussian process with variance

$$\mathrm{Var}(\hat{u}_j(t)) = \frac{\sigma^2 q_j}{2\lambda_j}(1 - e^{-2\lambda_j t}).$$

By the Parseval identity we obtain for $u_0 = 0$

$$\|u(t)\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbf{E}\left[\sum_{j=1}^{\infty} |\hat{u}_j(t)|^2\right] = \sum_{j=1}^{\infty} \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - e^{-2\lambda_j t}\right).$$

$$\|u(t)\|^2_{L^2(\Omega,L^2(0,\pi))} = \mathbf{E}\left[\sum_{j=1}^{\infty} |\hat{u}_j(t)|^2\right] = \sum_{j=1}^{\infty} \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - e^{-2\lambda_j t}\right).$$

- The series converges if the sum $\sum_{j=1}^{\infty} q_j/\lambda_j$ is finite.
  - For a Q-Wiener process, the sum is finite because Q is trace class. Hence solution $u(t)$ SHE is in $L^2(0,\pi)$ a.s.
  - For a cylindrical Wiener process, $q_j = 1$ and the sum is only finite if $\lambda_j \to \infty$ sufficiently quickly.
    We have , $\lambda_j = j^2$ and $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. Thus, $\|u(t)\|^2_{L^2(\Omega,L^2(0,\pi))} < \infty$. Hence solution $u(t) \in L^2(0,\pi)$ a.s.

SHE in two dimensions

Repeat the calculations with $D = (0, \pi) \times (0, \pi)$.
A has e.vals $\lambda_{j_1,j_2} = j_1^2 + j_2^2$ and normalised e.funcs ϕ_{j_1,j_2}, $j_1, j_2 \in \mathbb{N}$.
Assume that Q also has e.funcs ϕ_{j_1,j_2} and e.vals q_{j_1,j_2}.
Write $u(t) = \sum_{j_1,j_2=1}^{\infty} \hat{u}_{j_1,j_2}(t)\phi_{j_1,j_2}$.
Substituting $v = \phi_{j_1,j_2}$ into the weak form, each coefficient $\hat{u}_{j_1,j_2}(t)$ is an Ornstein–Uhlenbeck process:

$$d\hat{u}_{j_1,j_2} = -\lambda_{j_1,j_2}\hat{u}_{j_1,j_2}\,dt + \sigma\sqrt{q_{j_1,j_2}}\,d\beta_{j_1,j_2}(t)$$

and the variance

$$\mathrm{Var}(\hat{u}_{j_1,j_2}(t)) = \frac{\sigma^2 q_{j_1,j_2}}{2\lambda_{j_1,j_2}}\left(1 - e^{-2\lambda_{j_1,j_2}t}\right).$$

If $u_0 = 0$, $\mathbf{E}\left[\hat{u}_{j_1,j_2}(t)\right] = 0$ and

$$\|u(t)\|^2_{L^2(\Omega,L^2(D))} = \mathbf{E}\left[\sum_{j_1,j_2=1}^{\infty} |\hat{u}_{j_1,j_2}(t)|^2\right] = \sum_{j_1,j_2=1}^{\infty} \frac{\sigma^2 q_{j_1,j_2}}{2\lambda_{j_1,j_2}}\left(1 - e^{-2\lambda_{j_1,j_2}t}\right).$$

$$\|u(t)\|^2_{L^2(\Omega,L^2(D))} = \sum_{j_1,j_2=1}^{\infty} \frac{\sigma^2 q_{j_1,j_2}}{2\lambda_{j_1,j_2}} \left(1 - e^{-2\lambda_{j_1,j_2} t}\right).$$

- When Q is trace class, the right-hand side is finite. Solution $u(t) \in L^2(D)$ a.s.
- For a cylindrical Wiener process ($q_{j_1,j_2} \equiv 1$), we have

$$\sum_{j_1,j_2=1}^{\infty} \frac{1}{\lambda_{j_1,j_2}} = \sum_{j_1,j_2=1}^{\infty} \frac{1}{j_1^2 + j_2^2} = \infty$$

  and the solution $u(t)$ is not in $L^2(\Omega, L^2(D))$.

Do not expect weak solutions of SHE to exist in $L^2(D)$ in two dimensions.

▶ Need to take great care with cylindrical Wiener process !

Mild solution of $du = (-Au + f(u))dt + G(u)dW$

A predictable H-valued process $\{u(t)\colon t \in [0, T]\}$ is called a *mild solution* if for $t \in [0, T]$

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))\, ds + \int_0^t e^{-(t-s)A} G(u(s))\, dW(s),$$

where e^{-tA} is the semigroup generated by $-A$.

- Expect that all strong solutions are weak solutions.
- Expect all weak solutions are mild solutions.
- Reverse implications hold for solutions with sufficient regularity.

- Existence and uniqueness theory of mild solutions is easiest to develop.

In addition to the global Lipschitz condition on G, the following condition is used.

Assumption (Lipschitz condition on G)

For constants $\zeta \in (0,2]$ and $L > 0$, we have that $G\colon H \to L_0^2$ satisfies

$$\begin{aligned} \left\| A^{(\zeta-1)/2} G(u) \right\|_{L_0^2} &\leq L\big(1 + \|u\|\big), \\ \left\| A^{(\zeta-1)/2} \big(G(u) - G(v)\big) \right\|_{L_0^2} &\leq L \left\| u - v \right\|, \qquad \forall u, v \in H. \end{aligned} \tag{15}$$

For $\zeta > 1$, the operator $A^{(\zeta-1)/2}$ is unbounded
For $\zeta < 1$, it is smoothing
(because $A^{(\zeta-1)/2}\colon H \to \mathcal{D}(A^{\alpha}) \subset H$ for $\alpha = (1-\zeta)/2 > 0$).
Think $\zeta = 1$ - this is OK for Q Wiener process.

Existence and uniqueness

$$du = \Big[-Au + f(u)\Big]\,dt + G(u)\,dW(t), \qquad u(0) = u_0 \in H,$$

Suppose that A satisfies Assumption on linear operator.
$f\colon H \to H$ satisfies the global Lipschitz condition
$G\colon H \to L_0^2$ satisfies Assumption on noise.
Suppose that the initial data $u_0 \in L^2(\Omega, \mathcal{F}_0, L^2(D))$.
Then, there exists a unique mild solution $u(t)$

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))\,ds + \int_0^t e^{-(t-s)A} G(u(s))\,dW(s),$$

Furthermore, there exists a constant $K_T > 0$ such that

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(\Omega,H)} \le K_T\Big(1 + \|u_0\|_{L^2(\Omega,H)}\Big).$$

Proof: Standard fixed point argument.

Regularity additive noise

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s))\, ds + \int_0^t e^{-(t-s)A}\sigma\, dW(s),$$

Theorem (regularity in space for additive noise)

Let $G(u) = \sigma I$ and $\sigma \in \mathbb{R}$.
If $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{D}(A))$, then $u(t) \in L^2(\Omega, \mathcal{D}(A^{\zeta/2}))$ for $t \in [0, T]$.
So

$$\mathbf{E}\left[\|u(t)\|_{\zeta/2}\right] := \mathbf{E}\left[\left\|A^{\zeta/2}u(t)\right\|\right] < \infty$$

$\zeta = 1:$ Q-Wiener noise

Proof Split the mild solution into three terms, so that $u(t) = \mathtt{I} + \mathtt{II} + \mathtt{III}$, for

$$\mathtt{I} := e^{-tA}u_0,\ \mathtt{II} := \int_0^t e^{-(t-s)A} f(u(s))\, ds,\ \mathtt{III} := \int_0^t e^{-(t-s)A}\sigma\, dW(s).$$

- For the first term, since $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{D}(A))$, $\mathbf{E}\left[\|e^{-tA}u_0\|_{\zeta/2}^2\right] \le \mathbf{E}\left[\|u_0\|_1^2\right] < \infty$ and $\mathtt{I} \in L^2(\Omega, \mathcal{D}(A^{\zeta/2}))$.
- The second term also belongs to $L^2(\Omega, \mathcal{D}(A^{\zeta/2}))$.

$$\mathbf{E}\left[\|\mathtt{III}\|_{\zeta/2}^2\right] = \mathbf{E}\left[\left\|\int_0^t e^{-(t-s)A}\sigma\, dW(s)\right\|_{\zeta/2}^2\right]$$

For term $\mathtt{III}$, Itô's isometry gives

$$\mathbf{E}\left[\|\mathtt{III}\|_{\zeta/2}^2\right] = \sigma^2 \int_0^t \left\|A^{\zeta/2} e^{-(t-s)A}\right\|_{L_0^2}^2 \, ds.$$

Now,

$$\begin{aligned}\int_0^t \left\|A^{\zeta/2} e^{-(t-s)A}\right\|_{L_0^2}^2 \, ds &= \int_0^t \left\|A^{(\zeta-1)/2} A^{1/2} e^{-(t-s)A}\right\|_{L_0^2}^2 \, ds \\ &\leq \|A^{(\zeta-1)/2}\|_{L_0^2}^2 \int_0^t \|A^{1/2} e^{-(t-s)A}\|_{\mathcal{L}(H)}^2 \, ds.\end{aligned}$$

The integral is finite by standard semigroup results.
By Assumptions on G $\|A^{(\zeta-1)/2}\|_{L_0^2} < \infty$.
Hence, $\mathtt{III}$ belongs to $L^2(\Omega, \mathcal{D}(A^{\zeta/2}))$.

Reaction-diffusion equation, additive noise

Consider the SPDE

$$du = \Big[Au + f(u)\Big]\, dt + \sigma\, dW(t), \qquad u(0) = u_0 \in \mathcal{D}(A)$$

with $A = -u_{xx}$ and $\mathcal{D}(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$.
The operator A has eigenvalues $\lambda_j = j^2$.

- For Q-Wiener process, can take $\zeta = 1$ in Assumption 3 on G. By our additive noise Theorem 7, $u(t) \in L^2(\Omega, H^1(0,\pi))$. Our existence uniqueness only gave $L^2(0,\pi)$ spatial regularity.
- For space-time white noise (i.e., the cylindrical Wiener process), $\zeta \in (0, 1/2)$, because

$$\|A^{(\zeta-1)/2} G(u)\|_{L_0^2} = (\operatorname{Tr} A^{(\zeta-1)})^{1/2}$$

and $\lambda_j^{(\zeta-1)} = \mathcal{O}(j^{2(\zeta-1)})$.

- For the SHE in one dimension forced by space-time white noise takes values in $L^2(\Omega, H^\zeta(0,\pi))$ and has up to a half (generalised) derivatives almost surely.

Regularity in time

The exponents θ_1, θ_2 below determine rates of convergence for the numerical methods.
For simplicity assume $u_0 \in \mathcal{D}(A)$.
Eg $\zeta = 1$ for Lipschitz G.

Lemma (regularity in time)

For $T > 0$, $\epsilon \in (0, \zeta)$, and $\theta_1 := \min\{(\zeta - \epsilon)/2, 1/2\}$, there exists $K_{RT} > 0$ such that

$$\|u(t_2) - u(t_1)\|_{L^2(\Omega,H)} \le K_{RT}(t_2 - t_1)^{\theta_1}, \qquad 0 \le t_1 \le t_2 \le T. \tag{16}$$

Further, for $\theta_2 := (\zeta - \epsilon)/2$, there exits $K_{RT2} > 0$ such that

$$\left\| u(t_2) - u(t_1) - \int_{t_1}^{t_2} G(u(s))\, dW(s) \right\|_{L^2(\Omega,H)} \le K_{RT2}(t_2 - t_1)^{\theta_2}. \tag{17}$$

Proof. (Start)

Write $u(t_2) - u(t_1) = \mathtt{I} + \mathtt{II} + \mathtt{III}$, where

$$\mathtt{I} := \left(e^{-t_2 A} - e^{-t_1 A}\right) u_0, \ \mathtt{II} := \int_0^{t_2} e^{-(t_2 - s)A} f(u(s))\, ds - \int_0^{t_1} e^{-(t_1 - s)A} f(u(s))\, ds,$$

$$\mathtt{III} := \left(\int_0^{t_2} e^{-(t_2 - s)A} G(u(s))\, dW(s) - \int_0^{t_1} e^{-(t_1 - s)A} G(u(s))\, dW(s)\right).$$

The estimation of $\mathtt{I}$ and $\mathtt{II}$ like in a deterministic case, except the H norm replaced by the $L^2(\Omega, H)$ norm.
For $\mathtt{III}$ we write $\mathtt{III} = \mathtt{III}_1 + \mathtt{III}_2$, for

$$\mathtt{III}_1 := \int_0^{t_1} \left(e^{-(t_2 - s)A} - e^{-(t_1 - s)A}\right) G(u(s))\, dW(s),$$

$$\mathtt{III}_2 := \int_{t_1}^{t_2} e^{-(t_2 - s)A} G(u(s))\, dW(s).$$

Then use Itô isometry, assumption on G and standard estimates from semigroup theory ...
... for three pages.

Numerical methods

- We discretise in space : for example
  - Finite differences
  - Spectral Galerkin
  - Galerkin Finite element
- Discretise in time : for example
  - Euler–Maruyama
  - Milstein
- Strong convergence

Look at

$$\max_{0\leq t_n\leq T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \max_{0\leq t_n\leq T} \mathbf{E}\left[\|u(t_n) - \tilde{u}_n\|_H\right]$$

Finite difference method

Examine reaction-diffusion equation with additive noise

$$du = \left[\varepsilon\, u_{xx} + f(u)\right] dt + \sigma\, dW(t), \qquad u(0,x) = u_0(x), \qquad (18)$$

homogeneous Dirichlet boundary conditions on $(0,a)$.
$W(t)$ a Q-Wiener process on $L^2(0,a)$.

- Introduce the grid points $x_j = jh$ for $h = a/J$ and $j = 0,\ldots,J$.

Use centred difference approximation $A^{\mathrm{D}} \approx \Delta$

$$A^{\mathrm{D}} := \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix},$$

$\mathbf{u}_J(t) \approx [u(t,x_1),\ldots,u(t,x_{J-1})]^T$ solves

$$d\mathbf{u}_J = \left[-\varepsilon\, A^{\mathrm{D}}\mathbf{u}_J + \mathbf{f}(\mathbf{u}_J)\right] dt + \sigma\, d\mathbf{W}_J(t).$$

$\mathbf{W}_J(t) := [W(t,x_1),\ldots,\ W(t,x_{J-1})]^T.$

Discretise in time :

Methods : Euler–Maruyama, Milstein etc

We examine semi-implicit Euler–Maruyama method wtih time step $\Delta t > 0$

- This has good stability properties

Get approximation $\mathbf{u}_{J,n}$ to $\mathbf{u}_J(t_n)$ at $t_n = n\Delta t$

$$\mathbf{u}_{J,n+1} = \left(I + \Delta t\, \varepsilon\, A^{\mathrm{D}}\right)^{-1} \left[\mathbf{u}_{J,n} + \mathbf{f}(\mathbf{u}_{J,n})\, \Delta t + \sigma\, \Delta \mathbf{W}_n\right]$$

with $\mathbf{u}_{J,0} = \mathbf{u}_J(0)$ and $\Delta \mathbf{W}_n := \mathbf{W}_J(t_{n+1}) - \mathbf{W}_J(t_n)$.

$\mathbf{W}_J(t) := [W(t, x_1), \ldots,\ W(t, x_{J-1})]^T$.

Space-time white noise

The covariance $Q = I$

- Derive an approximation to the increment $W(t_{n+1}) - W(t_n)$.
- Truncate the expansion of $W(t)$ to J terms.

Take as basis $\{\sqrt{2/a}\sin(j\pi x/a)\}$ of $L^2(0,a)$

$$W^J(t,x) = \sqrt{2/a}\sum_{j=1}^{J} \sin\left(\frac{j\pi x}{a}\right)\beta_j(t),$$

for *iid* Brownian motions $\beta_j(t)$.

- $\mathrm{Cov}(W^J(t,x_i), W^J(t,x_k)) = \mathbf{E}\big[W^J(t,x_i)W^J(t,x_k)\big]$

$$= \frac{2\,t}{a}\sum_{j=1}^{J}\sin\left(\frac{j\pi x_i}{a}\right)\sin\left(\frac{j\pi x_k}{a}\right).$$

Using $x_i = ih$ and $h = a/J$ with a trigonometric identity gives

$$2\sin\left(\frac{j\pi x_i}{a}\right)\sin\left(\frac{j\pi x_k}{a}\right) = \cos\left(\frac{j\pi(i-k)}{J}\right) - \cos\left(\frac{j\pi(i+k)}{J}\right).$$

Now,

$$\sum_{j=1}^{J} \cos\left(\frac{j\pi m}{J}\right) = \begin{cases} J, & m = 0, \\ 0, & m \text{ even and } m \neq 0, \\ -1, & m \text{ odd.} \end{cases}$$

Therefore,

$$\mathrm{Cov}(W^J(t, x_i), W^J(t, x_k)) = \frac{2\,t}{a} \sum_{j=1}^{J} \sin\left(\frac{j\pi x_i}{a}\right) \sin\left(\frac{j\pi x_k}{a}\right)$$

becomes

$$\mathrm{Cov}(W^J(t, x_i), W^J(t, x_k)) = (t/h)\,\delta_{ik}$$

for $i, k = 1, \ldots, J$.

We now use $W^J(t)$ when $W(t)$ is space-time white noise.

Spatial Approx. Reaction-Diffusion equation by

$$d\mathbf{u}_J = \left[-\varepsilon\, A^{\mathrm{D}} \mathbf{u}_J + \mathbf{f}(\mathbf{u}_J)\right] dt + \sigma\, d\mathbf{W}^J(t)$$

for $\mathbf{W}^J(t) := [W^J(t, x_1), \ldots, W^J(t, x_{J-1})]^T$.

And have $\mathbf{W}^J(t) \sim N(\mathbf{0}, (t/h)\, I)$.

Discretise in time

$$d\mathbf{u}_J = \left[-\varepsilon\, A^{\mathrm{D}}\mathbf{u}_J + \mathbf{f}(\mathbf{u}_J)\right] dt + \sigma\, d\mathbf{W}^J(t)$$

$\mathbf{W}^J(t) \sim N(\mathbf{0}, (t/h)\, I)$.

For a time step $\Delta t > 0$, the semi-implicit Euler–Maruyama method gives

$$\mathbf{u}_{J,n+1} = (I + \varepsilon\, A^{\mathrm{D}} \Delta t)^{-1} \left[\mathbf{u}_{J,n} + \Delta t\, \mathbf{f}(\mathbf{u}_{J,n}) + \sigma \Delta\mathbf{W}_n\right]$$

and $\Delta\mathbf{W}_n \sim N(\mathbf{0}, (\Delta t/h) I)$ *iid*.

```
function [t,ut]=spde_fd_d_white(u0,T,a,N,J,epsilon,sigma,fh
Dt=T/N;   t=[0:Dt:T]'; h=a/J;
% set matrices
e = ones(J+1,1);    A = spdiags([e -2*e e], -1:1, J+1, J+1)
%case {'dirichlet','d'}
ind=2:J;   A=A(ind,ind);
EE=speye(length(ind))-Dt*epsilon*A/h/h;
ut=zeros(J+1,length(t)); % initialize vectors
ut(:,1)=u0; u_n=u0(ind); % set initial condition
for k=1:N, % time loop
  fu=fhandle(u_n); Wn=sqrt(Dt/h)*randn(J-1,1);
  u_new=EE\(u_n+Dt*fu+sigma*Wn);
  ut(ind,k+1)=u_new; u_n=u_new;
end
```

Code to generate realisations of the finite difference approximation
homogeneous Dirichlet boundary conditions
space-time white noise.

Galerkin approximation

Based on weak solution

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \Big[-\langle u(s), Av\rangle + \langle f(u(s)), v\rangle\Big]\, ds + \int_0^t \langle G(u(s))\, dW(s), v\rangle,$$

where

$$\int_0^t \langle G(u(s))\, dW(s), v\rangle := \sum_{j=1}^{\infty} \int_0^t \Big\langle G(u(s))\sqrt{q_j}\,\chi_j, v\Big\rangle\, d\beta_j(s).$$

- Take finite-dimensional subspace
$\tilde V = \mathrm{span}\{\psi_1, \psi_2, \dots, \psi_J\} \subset \mathcal{D}(A^{1/2})$.
Let $\tilde P$ be the orthogonal projection $\tilde P : H \to \tilde V$
Seek $u(t) \approx \tilde u(t) = \sum_{j=1}^{J} \hat u_j(t)\psi_j$
Initial data, we take $\tilde u_0 = \tilde P\, u_0$
Rewrite as

$$d\tilde u = \Big[-\tilde A \tilde u + \tilde P\, f(\tilde u)\Big]\, dt + \tilde P\, G(\tilde u)\, dW(t), \qquad \tilde u(0) = \tilde u_0,$$

where $\langle \tilde A w, v\rangle = \langle A^{1/2} w, A^{1/2} v\rangle$.
- Discretise in time

$$\tilde u_{n+1} = \big(I + \Delta t \tilde A\big)^{-1}\Big(\tilde u_n + \tilde P\, f(\tilde u_n)\,\Delta t + \tilde P\, G(\tilde u_n)\,\Delta W_n\Big)$$

$$\tilde{u}_{n+1} = \left(I + \Delta t \tilde{A}\right)^{-1} \left(\tilde{u}_n + \tilde{P}\, f(\tilde{u}_n)\, \Delta t + \tilde{P}\, G(\tilde{u}_n)\, \Delta W_n \right)$$

for $\Delta W_n := \int_{t_n}^{t_{n+1}} dW(s)$.

- In practice, it is necessary to approximate G with some $\mathcal{G} \colon \mathbb{R}^+ \times H \to L_0^2$

$$\tilde{u}_{n+1} = \left(I + \Delta t \tilde{A}\right)^{-1} \left(\tilde{u}_n + \tilde{P}\, f(\tilde{u}_n)\, \Delta t + \tilde{P} \int_{t_n}^{t_{n+1}} \mathcal{G}(s; \tilde{u}_n)\, dW(s) \right),$$

- Example : $\mathcal{G}(s; u) = G(u)$

- $\mathcal{G}(s; u)$ acts on the infinite-dimensional U-valued process $W(t)$. Difficult to implement as a numerical method.

Usually consider $\mathcal{G}(s; u) = G(u)\mathcal{P}_{J_w}$ for the orthogonal projection $\mathcal{P}_{J_w} \colon U \to \operatorname{span}\{\chi_1, \dots, \chi_{J_w}\}$ given an orthonormal basis χ_j of U.

δ : spatial discretisation parameter (e.g. $\delta = h$).

Assumption

For some $\zeta \in (0,2]$, let Assumption on G hold and, for some constants $K_{\mathcal{G}}, \theta, L > 0$, let $\mathcal{G} \colon \mathbb{R}^+ \times H \to L_0^2$ satisfy

$$\left\| \mathcal{G}(s; u_1) - \mathcal{G}(s; u_2) \right\|_{L_0^2} \leq L \left\| u_1 - u_2 \right\|, \qquad \forall s > 0,\ u_1, u_2 \in H, \tag{19}$$

and for $t_k \leq s < t_{k+1}$

$$\left\| \tilde{P}\Big(G(u(s)) - \mathcal{G}(s; u(t_k)) \Big) \right\|_{L^2(\Omega, L_0^2)} \leq K_{\mathcal{G}} \Big(|s - t_k|^{\theta} + \delta^{\zeta} \Big). \tag{20}$$

This assumption holds for $\mathcal{G}(s,u) \equiv \mathcal{G}(u) := G(u)\mathcal{P}_{J_w}$ for a broad class of Q-Wiener processes.

Under set of conditions on the Galerkin subspace $\tilde{V}$, we prove strong convergence.

Theorem (strong convergence)

Let the following assumptions hold:

1. *the Assumptions for unique mild solution.*
2. *the initial data* $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{D}(A))$.
3. *Suppose that* $\tilde{A}^{-1} \in \mathcal{L}(H)$ *satisfies* $\tilde{A}^{-1}\tilde{A} = I$ *on* $\tilde{V}$ *and* $\tilde{A}^{-1}(I - \tilde{P}) = 0$ *and is non-negative definite. Further, for some* $C, \delta > 0$

$$\left\| \left(\tilde{A}^{-1} - A^{-1} \right) f \right\| \leq C\, \delta^2 \, \| f \| \,, \qquad \forall f \in H$$

4. *Assumption on* $\mathcal{G}$ *for some* $\theta > 0$ *and* $\zeta \in (0, 2]$.

If $\Delta t / \delta^2$ *is fixed, then for each* $\epsilon > 0$, *there exists* $K > 0$ *such that*

$$\max_{0 \leq t_n \leq T} \| u(t_n) - \tilde{u}_n \|_{L^2(\Omega, H)} \leq K \left(\Delta t^{(\zeta - \epsilon)/2} + \Delta t^{\theta} \right).$$

Proof: Assume without loss of generality that $\Delta t/\delta^2 = 1$.
Using the notation $\tilde{S}_{\Delta t} := (I + \Delta t \tilde{A})^{-1}$,
Scheme after n steps :

$$\tilde{u}_n = \tilde{S}_{\Delta t}^n \tilde{P} u_0 + \sum_{k=0}^{n-1} \tilde{S}_{\Delta t}^{n-k} \tilde{P} f(\tilde{u}_k) \Delta t + \sum_{k=0}^{n-1} \tilde{S}_{\Delta t}^{n-k} \tilde{P} \int_{t_k}^{t_{k+1}} \mathcal{G}(s, \tilde{u}_k) \, dW(s).$$

Subtracting from the mild solution (14),
$u(t_n) - \tilde{u}_n = \mathtt{I} + \mathtt{II} + \mathtt{III}$ for

$$\mathtt{I} := \left(e^{-t_n A} u_0 - \tilde{S}_{\Delta t}^n \tilde{P} u_0 \right),$$

$$\mathtt{II} := \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} e^{-(t_n - s)A} P f(u(s)) \, ds - \tilde{S}_{\Delta t}^{n-k} \tilde{P} f(\tilde{u}_k) \Delta t \right),$$

$$\mathtt{III} := \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left(e^{-(t_n - s)A} G(u(s)) - \tilde{S}_{\Delta t}^{n-k} \tilde{P} \mathcal{G}(s, \tilde{u}_k) \right) dW(s).$$

To treat $\mathtt{I}$ and $\mathtt{II}$: like deterministic case.

$$\| \mathtt{I} + \mathtt{II} \|_{L^2(\Omega, H)} \le C_{\mathtt{I}+\mathtt{II}} (\Delta t + \delta^2) \Delta t^{-\epsilon}$$

for a constant $C_{\mathtt{I}+\mathtt{II}}$.

We break $\texttt{III}$ into four further parts by writing

$$e^{-(t_n-s)A}G(u(s)) - \tilde{S}_{\Delta t}^{n-k}\tilde{P}\,\mathcal{G}(s,\tilde{u}_k) = X_1 + X_2 + X_3 + X_4$$

for

$$X_1 := \Big(e^{-(t_n-s)A} - e^{-(t_n-t_k)A}\Big)G(u(s)), \quad X_2 := \Big(e^{-(t_n-t_k)A} - \tilde{S}_{\Delta t}^{n-k}\tilde{P}\Big)G(u(s)),$$
$$X_3 := \tilde{S}_{\Delta t}^{n-k}\tilde{P}\Big(G(u(s)) - \mathcal{G}(s;u(t_k))\Big), \quad X_4 := \tilde{S}_{\Delta t}^{n-k}\tilde{P}\Big(\mathcal{G}(s;u(t_k)) - \mathcal{G}(s;\tilde{u}_k)\Big).$$

- To estimate $\texttt{III}$ in $L^2(\Omega, H)$, we estimate $\texttt{III}_i = \int_0^{t_n} X_i\, dW(s)$ separately using the triangle inequality.
  Use Itô's isometry and estimates from semigroup theory and Gronwall.

Example (reaction-diffusion equation on $(0,1)$)

$$du = \Big[-Au + f(u)\Big]\,dt + G(u)\,dW(t),$$

where $A = -\Delta$ with $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$

- $W(t)$ a Q-Wiener process.

If $G(u)W(t)$ is smooth $\zeta = 1$,
Choose $\mathcal{G}(u) = G(u)\mathcal{P}_{J_w}$ with J_w sufficiently large .
For initial data $u_0 \in H^2(0,1) \cap H_0^1(0,1)$, have

$$\max_{0\le t_n \le T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \mathcal{O}(\Delta t^{1/2} + \delta).$$

- Additive noise : improved rate convergence

$$\max_{0\le t_n \le T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \mathcal{O}(\Delta t^{1-\epsilon} + \delta).$$

- For additive space-time white noise, $W(t)$ cylindrical Wiener process, $\zeta \in (0,1/2)$ and

$$\max_{0\le t_n \le T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \mathcal{O}(\Delta t^{(1-\epsilon)/4} + \Delta t^{\theta}), \qquad \epsilon > 0.$$

Spectral Galerkin

$$du = \Big[-Au + f(u)\Big]\,dt + G(u)\,dW(t)$$

periodic boundary conditions on the domains $D = (0, a)$
Approximate using the Galerkin subspace
$\tilde{V} = V_J := \mathrm{span}\{\phi_1, \ldots, \phi_J\}$
ϕ_j eigenfunctions of A.

$$P_J u := \sum_{j=1}^{J} \hat{u}_j \phi_j, \qquad \hat{u}_j := \frac{1}{\|\phi_j\|^2}\big\langle u, \phi_j \big\rangle, \quad u \in H.$$

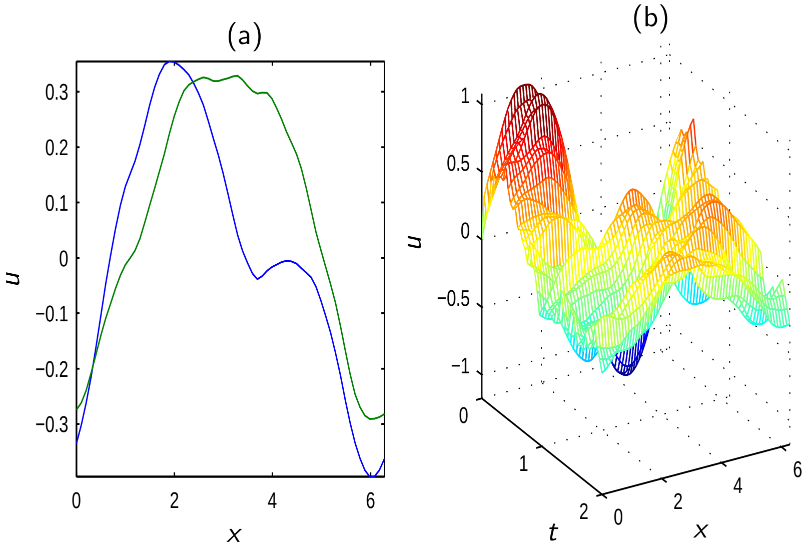
- Spatial discretisation. $P_J = P_{J_w}$.

$$du_J = \Big[-A_J u_J + P_J\, f(u_J)\Big]\,dt + P_J\, G(u_J)\,dW(t), \qquad u_J(0) = P_J u_0$$

- Time discretisation

$$u_{J,n+1} = (I + \Delta t\; A_J)^{-1}\Big(u_{J,n} + \Delta t\, P_J\, f(u_{J,n}) + P_J\, G(u)\mathcal{P}_{J_w}\,\Delta W_n\Big).$$

Allen Cahn : $du = (u_{xx} + u - u^3)dt + dW$.



```
function [t,u,ut]=spde_AC(u0,T,a,N,Jref,r,sigma)
Dt=T/N; t=[0:Dt:T]';
% set Lin Operators
kk  = 2*pi*[0:Jref/2 -Jref/2+1:-1]'/a;
Dx = (1i*kk); MM=-Dx.^2;
EE=1./(1+Dt*MM);
% get form of noise
iFspace=1; bj = get_oned_bj(Dt,Jref,a,r);
% set initial condition
ut(:,1)=u0;  u=u0(1:Jref); uh0=fft(u); uh=uh0;
u=real(ifft(uh));
for n=1:N % time loop
  fhu=fft(u-u.^3);
  dW=get_oned_dW(bj,1,iFspace,1);
  gu=sigma;     % function for noise term
  gdWh=fft(gu.*real(ifft(dW))); %
  uh_new=EE.*(uh+Dt*fhu+gdWh);
  uh=uh_new;
  u=real(ifft(uh));
  ut(1:Jref,n+1)=u(:,1);
end
ut(Jref+1,:)=ut(1,:); u=[u; u(1,:)]; % periodic
```

Convergence

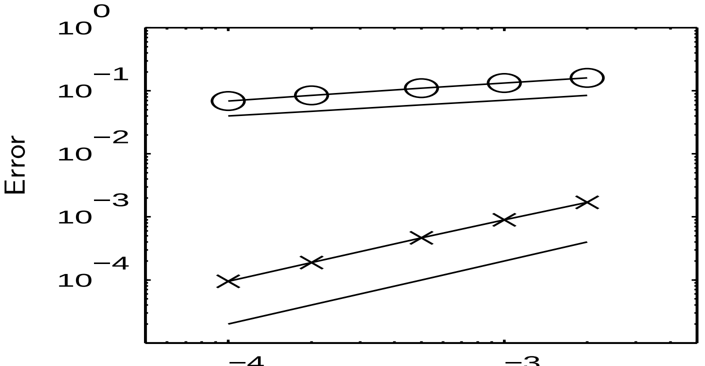
Allen Cahn : $du = (u_{xx} + u - u^3)dt + dW$.

- Additive noise : improved rate convergence

$$\max_{0 \leq t_n \leq T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \mathcal{O}(\Delta t^{1-\epsilon} + \delta).$$

- For additive space-time white noise, $W(t)$ cylindrical Wiener process, $\zeta \in (0, 1/2)$ and

$$\max_{0 \leq t_n \leq T} \|u(t_n) - \tilde{u}_n\|_{L^2(\Omega,H)} = \mathcal{O}(\Delta t^{(1-\epsilon)/4} + \Delta t^{\theta}), \qquad \epsilon > 0.$$



Galerkin Finite Element

$$du = \Big[\varepsilon\,\Delta u + f(u)\Big]\,dt + g(u)\,dW(t), \qquad u(0) = u_0 \in L^2(D)$$

Let $\tilde{V} = V^h =$ space of continuous and piecewise linear functions.
Take uniform mesh of n_e elements with vertices
$0 = x_0 < \cdots < x_{n_e} = a$. mesh width $h = a/n_e$.
Finite element approximation $u_h(t) \in V^h$

$$u_h(t,x) = \sum_{j=1}^{J} u_j(t)\,\phi_j(x).$$

- Space discretisation

$$du_h = \Big[-\varepsilon\,A_h u_h + P_{h,L^2}\,f(u_h)\Big]\,dt + P_{h,L^2}\,G(u_h)\,dW(t)$$

where A_h is defined by $\big\langle A_h w, v\big\rangle = a(w,v)$.

- Time discretisation

$$u_{h,n+1} = \big(I + \Delta t\,\varepsilon\,A_h\big)^{-1}\Big(u_{h,n} + P_{h,L^2}\,f(u_{h,n})\,\Delta t + P_{h,L^2}\,\mathcal{G}(u_{h,n})\,\Delta W_n\Big)$$

Equ. for coefficients

$$u_h(t,x) = \sum_{j=1}^{J} u_j(t)\,\phi_j(x)$$

Note that $\mathcal{P}_{J_w}\colon U \to \text{span}\{\chi_1,\dots,\chi_{J_w}\}$ and $P_J\colon H \to V^h$.
Distinct operators.

- Let $\mathbf{u}_h(t) := [u_1(t), u_2(t), \dots, u_J(t)]^T$. Then, we get

$$M d\mathbf{u}_h = \Big[-\varepsilon K\mathbf{u}_h + \mathbf{f}(\mathbf{u}_h)\Big]\,dt + \mathbf{G}(\mathbf{u}_h)\,dW(t),$$

$\mathbf{f}(\mathbf{u}_h) \in \mathbb{R}^J$ has elements $f_j = \langle f(u_h), \phi_j\rangle_{L^2(0,a)}$.
M is the mass matrix with elements $m_{ij} = \langle \phi_i, \phi_j\rangle_{L^2(0,a)}$
K is the diffusion matrix with elements $k_{ij} = a(\phi_i, \phi_j)$.
Finally, $\mathbf{G}\colon \mathbb{R}^J \to \mathcal{L}(U, \mathbb{R}^J)$
and $\mathbf{G}(\mathbf{u}_h)\chi$ has jth coefficient

$$\big\langle G(u_h)\chi, \phi_j\big\rangle_{L^2(0,a)}$$

for $\chi \in U$.

Time discrete

$$M d\mathbf{u}_h = \Big[-\varepsilon\, K\mathbf{u}_h + \mathbf{f}(\mathbf{u}_h)\Big]\, dt + \mathbf{G}(\mathbf{u}_h)\, dW(t),$$

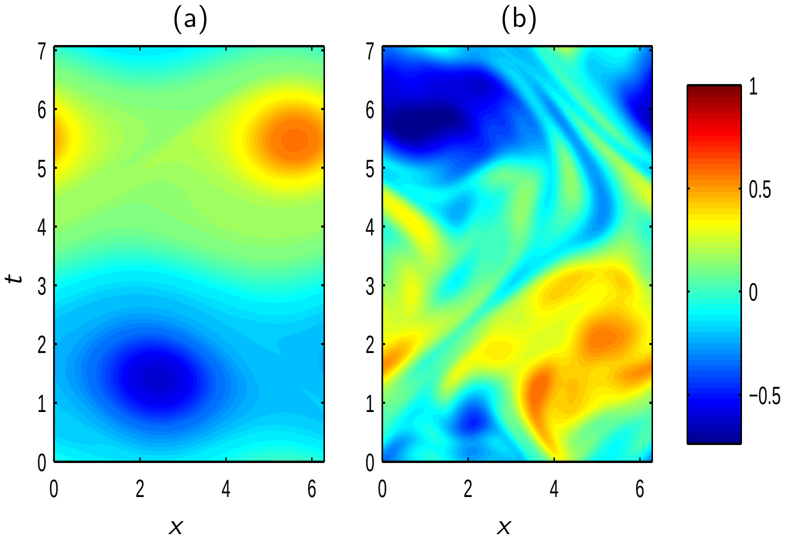
- Use semi–implicit Euler–Maruyama

$$\big(M + \Delta t\, \varepsilon\, K\big)\mathbf{u}_{h,n+1} = M\,\mathbf{u}_{h,n} + \Delta t\, \mathbf{f}(\mathbf{u}_{h,n}) + G_h(\mathbf{u}_{h,n})\, \Delta\mathbf{W}_n$$

- The term $G_h(\mathbf{u}_{h,n}) \in \mathbb{R}^{J\times J_w}$ has j,k entry $\big\langle G(\mathbf{u}_{h,n})\chi_k, \phi_j\big\rangle_{L^2(0,a)}$
- Term $\Delta\mathbf{W}_n$ is a vector in $\mathbb{R}^{J_w}$ with entries $\big\langle W(t_{n+1}) - W(t_n), \chi_k\big\rangle_{L^2(0,a)}$ for $k = 1, \ldots, J_w$.
- Practical computations:
Write the Q-Wiener process $W(t)$ as series.
Then $G_h(\mathbf{u}_{h,n})\Delta\mathbf{W}_n$ is found by multiplying the matrix G_h by the vector of coefficients

$$[\sqrt{q_1}(\beta_1(t_{n+1}) - \beta_1(t_n)), \ldots, \sqrt{q_{J_w}}(\beta_{J_w}(t_{n+1}) - \beta_{J_w}(t_n))]^T.$$

Stochastic Navier Stokes:



(a) Q-Wiener process $W(t)$ in $H_0^1(0,1)$

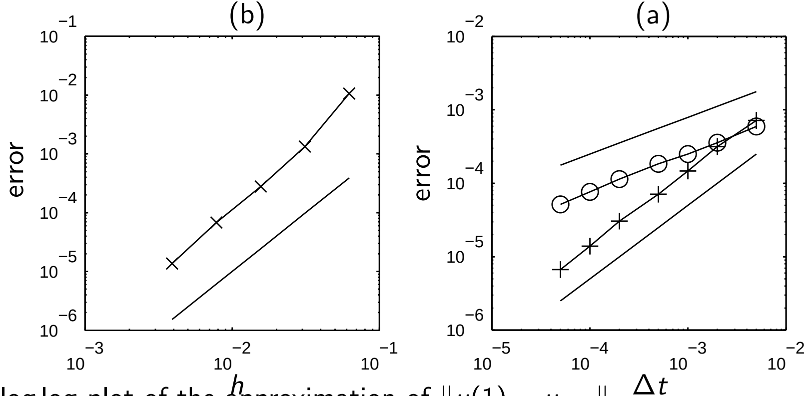
(b) space-time white noise $(H_0^{1/2}(0,1))$.

Numerical Convergence

We approximate

$$\|u(T) - u_{h,N}\|_{L^2(\Omega, L^2(0,a))} \approx \left(\frac{1}{M} \sum_{m=1}^{M} \|u_{\text{ref}}^m - u_{h,N}^m\|_{L^2(0,a)}^2 \right)^{1/2}. \tag{21}$$

Finite element and semi-implicit Euler approximation of the stochastic Nagumo equation.



log log plot of the approximation of $\|u(1) - u_{h,N}\|_{L^2(\Omega, L^2(0,a))}$
(a) the spatial mesh size h is varied and
(b) as the time step Δt is varied.
Multiplicative noise gives errors of order $\Delta t^{1/2}$
Additive noise gives errors of order Δt.

Exponential integrator for additive noise

The semi-implicit Euler–Maruyama method uses a basic increment ΔW_n to approximate $W(t)$.
An alternative time stepping method :
use the mild solution/ variation of constants formula for SPDEs.

$$u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A} f(u(s))ds + \int_0^t e^{(t-s)A} g(u(s))dW(s).$$

Consider discretization in space via : $u_J(t) = \sum_{j=1}^{J} \hat{u}_j(t)\phi_j$.
The variation of constants formula in each mode with $t_n = n\Delta t$

$$\hat{u}_j(t_{n+1}) = e^{-\Delta t \lambda_j} \hat{u}_j(t_n) + \int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-s)\lambda_j} \hat{f}_j(u_J(s))\, ds$$

$$+\sigma \int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-s)\lambda_j} \sqrt{q_j}\, d\beta_j(s).$$

To obtain a numerical method, we approximate $\hat{f}_j(u_J(s))$ by $\hat{f}_j(u_J(t_n))$ for $s \in [t_n, t_{n+1})$ and evaluate the integral, to find

$$\int_{t_n}^{t_{n+1}} e^{-(t_{n+1}-s)\lambda_j} \hat{f}_j(u_J(s))\, ds \approx \frac{1 - e^{-\Delta t \lambda_j}}{\lambda_j} \hat{f}_j(u_J(t_n)).$$

For the stochastic integral, we usually approximate $e^{-(t_{n+1}-s)\lambda_j} \approx e^{-t_{n+1}\lambda_j}$ and use a standard Brownian increment. However,

$$\mathbf{E}\left[\left|\int_0^t e^{-s\lambda} d\beta_j(s)\right|^2\right] = \frac{1 - e^{-2t\lambda}}{2\lambda}.$$

The stochastic integral $\int_0^t e^{-s\lambda} d\beta_j(s)$ has distribution $N(0, (1 - e^{-2t\lambda})/2\lambda)$.

Hence can generate approximations $\hat{u}_{j,n}$ to $\hat{u}_j(t_n)$ using

$$\hat{u}_{j,n+1} = e^{-\Delta t \lambda_j} \hat{u}_{n,j} + \frac{1 - e^{-\Delta t \lambda_j}}{\lambda_j} \hat{f}_j(u_{J,n}) + \sigma\, b_j\, R_{j,n} \qquad (22)$$

where $b_j := \sqrt{q_j(1 - e^{-2\Delta t \lambda_j})/2\lambda_j}$ and $R_{j,n} \sim N(0,1)$ *iid*.

- Advantage : samples the stochastic integral term exactly.