Frobenius-Stickelberger Formulae for General Curves

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The original Frobenius-Stickelberger formula is the following equation satisfied by the Weierstrass functions $\sigma(u)$ and $\wp(u) = -\frac{d^2}{du^2}\log\sigma(u)$:

$$\frac{\sigma(u^{(1)}+\cdots+u^{(n)})\prod_{i<j}\sigma(u^{(i)}-u^{(j)})}{\prod_{j=1}^{n}\sigma(u^{(j)})^n}$$
$$= \frac{(-1)^{(n-1)(n-2)/2}}{1!\,2!\,3!\cdots(n-1)!}\cdot\begin{vmatrix} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \wp''(u^{(1)}) & \wp^{(3)}(u^{(1)}) & \cdots \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \wp''(u^{(2)}) & \wp^{(3)}(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \wp''(u^{(n)}) & \wp^{(3)}(u^{(n)}) & \cdots \end{vmatrix},$$

($n \times n$ determinant).

(Modification)

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For $\mathscr{C}: y^2 + (\mu_1 x + \mu_3) y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$, it holds that

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\prod_{j=1}^{n} \sigma(u^{(j)})^n}$$
$$= (-1)^n \begin{vmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & xy(u^{(1)}) & x^3(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & xy(u^{(2)}) & x^3(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & xy(u^{(n)}) & x^3(u^{(n)}) & \cdots \end{vmatrix},$$

($n \times n$ determinant), where $x(u)$, $y(u)$ is just x, y determined by

$$u = \int_{\infty}^{(x,y)} \frac{\mathrm{d}x}{2y + \mu_1 x + \mu_3}.$$

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We denote in decreasing order

$\{w_g,\ w_{g-1},\ \cdots,\ w_1\} = \mathbb{Z}_{\geq 0} \setminus \{ad + bq \mid a \geq 0,\ b \geq 0\}$

(the Weierstrass gap sequence w.r.t. (d, q)).

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Example 3. $(d, q) = (3, 4)$

$$y^3 + (\mu_1 x + \mu_5)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y$$
$$= x^4 + \mu_3 x^2 + \mu_6 x + \cdots + \mu_{12}$$

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For $u^{(i)} \bmod \Lambda \in W^{[1]}$ $(1 \leqq i \leqq n)$, the following euqality holds:

$$\frac{\sigma_{\natural^n}(u^{(1)}+u^{(2)}+\cdots+u^{(n)}) \prod_{i<j} \prod_{\substack{\gamma\in\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)\\ \gamma\neq\mathbf{id}}} \sigma_\flat(u^{(i)}+[\gamma]u^{(j)})}{\prod_{j=1}^{n}\left(\sigma_\sharp(u^{(j)})^{(d-1)(n-j)+1} \prod_{\substack{\gamma\in\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)\\ \gamma\neq\mathbf{id}}} \sigma_\sharp([\gamma]u^{(j)})^{j-1}\right)}$$

$$= \pm \left| (x^{a_j}y^{b_j})(u^{(i)}) \right|_{1\leqq i,j\leqq n} \cdot \left| (x^{j-1})(u^{(i)}) \right|_{1\leqq i,j\leqq n}^{d-2},$$

where $\{da_j+qb_j\}$ is the Weierstrass non-gap sequence at ∞.

Theorem. This is OK for $(d,q)=(2,$ "any"$)$, $(3,4)$, $(3,5)$, $(4,5)$, $(5,6)$.

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For example, the equation of $\mathscr{C}$ in case of $(d, q) = (3, 4)$,

$$y^3 + (\mu_1 x + \mu_5) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y$$
$$= x^4 + \mu_3 x^2 + \mu_6 x + \cdots + \mu_{12}$$

is homogeneous of weight $d \cdot q = 3 \times 4 = 12$.

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where $t = -x/y$ (*the arithmetic parameter*)

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$$H^1(\mathscr{C}, \mathbb{C}) \cong \frac{H^0\big(\mathscr{C}, \mathbf{d}\mathscr{O}(*\infty)\big)}{\mathbf{d}H^0\big(\mathscr{C}, \mathscr{O}(*\infty)\big)}$$ (by Serre duality, etc.)

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For any ω and η in this space, we define

$$\omega \star \eta = \frac{1}{2\pi}\int_{\mathscr{C}} \omega \wedge {}^{*}\eta = \frac{1}{2\pi i}\int_{\partial\mathscr{C}_{\mathrm{r.p.}}} \Big(\int_{\infty}^{\mathrm{P}} \omega\Big)\eta(\mathrm{P}) = \sum_{\mathrm{P}\in\mathscr{C}} \operatorname*{Res}_{\mathrm{P}} \Big(\int_{\infty}^{\mathrm{P}} \omega\Big)\eta(\mathrm{P})$$

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$$= \frac{1}{2\pi i} \sum_{j=1}^{g} \Big(\int_{\alpha_j} \omega \int_{\beta_j} \eta - \int_{\alpha_j} \eta \int_{\beta_j} \omega \Big).$$

where $\{ \alpha_j, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \}$ is a symplectic base of $H_1(\mathscr{C}, \mathbb{Z})$.

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where $\{ \alpha_j, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \}$ is a symplectic base of $H_1(\mathscr{C}, \mathbb{Z})$.

This product is just the transported one from the usual symplectic structure on $H_1(\mathscr{C}, \mathbb{Z}) \otimes \mathbb{C}$ under $H^1(\mathscr{C}, \mathbb{C}) \cong H^1(\mathscr{C}, \mathbb{C})^\vee \cong H_1(\mathscr{C}, \mathbb{Z}) \otimes \mathbb{C}$.

Note that $\omega_i \star \omega_j = 0$. We extend $\{\,\omega_1,\ \cdots,\ \omega_g\,\}$ to a symplectic base

$$\{\,\omega_1,\ \cdots,\ \omega_g,\ \eta_1,\ \cdots,\ \eta_g\,\}$$

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the 2-form (*Klein's fundamental 2-form*) defined by

$$\xi(x,y;z,w) =$$
$$\omega_1(x,y) \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(x-z)} \frac{f(Z,y) - f(Z,w)}{y-w} \Big|_{Z=z} \mathrm{d}z - \sum_{j=1}^{g} \omega_j(x,y)\, \eta_j(z,w),$$

is symmetric, i.e. $\xi(x,y;z,w) = \xi(z,w;x,y)$, and

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$$\xi(x, y; z, w) \in \frac{1}{(t_2 - t_1)^2} + \mathbb{Z}[\mu][[t_1, t_2]],$$

where t_1 and t_2 are the arithmetic local parameter of (x, y) and (z, w) on $\mathscr{C}$, respectively.

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Choice of $\{\eta_j\}$ is not unique, but we chose the "simplest" one.

Example 1. $y^2 + (\mu_1 x + \mu_3) y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$, $(g = 1)$.

$$\omega_1 = \frac{dx}{f_y(x, y)} \in (1 + t\,\mathbb{Z}[\mu][[t]])dt,$$.

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.

$$\xi = \frac{F(x,y;z,w)dxdz}{(x-z)^2 f_y(x,y) f_y(z,w)},$$

where

$$F(x,y;z,w) = xz(x+z) + ({\mu_1}^2 + 2\mu_2)xz + \mu_1(zy + xw) + (\mu_3\mu_1 + \mu_4)(x+z) + 2yw + \mu_3(y+w) + {\mu_3}^2 + 2\mu_6$$

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$$+ (\mu_3\mu_1 + \mu_4)(x+z) + 2yw + \mu_3(y+w) + {\mu_3}^2 + 2\mu_6$$

Then

$$\eta_1 = \frac{x\mathbf{d}x}{f_y(x,y)} \in (t^2 + t^3\,\mathbb{Z}[\mu][[t]])\mathbf{d}t.$$

Example 2. For $y^2 + (\mu_1 x + \mu_3) y = x^5 + \mu_2 x^4 + \cdots + \mu_{10}$, $(g = 2)$,

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$$\omega_1 = \frac{\mathbf{d}x}{f_y(x,y)}, \quad \omega_2 = \frac{x\mathbf{d}x}{f_y(x,y)},$$

we have $\quad \xi(x,y;z,w) = \dfrac{F(x,y;z,w)dx_1dz}{(z-x)^2 f_y(x_1,y_1) f_w(z,w)}$, where

$$F(x,y;z,w) = (x^2z^3+x^3z^2)+({\mu_1}^2+2\mu_2)x^2z^2+(\mu_3\mu_1+\mu_4)(xz^2+x^2z)$$
$$+\mu_1(yz^2+wx^2)+(2\mu_5\mu_1+{\mu_3}^2+2\mu_6)xz+\mu_3(yz+wx)$$
$$+(\mu_5\mu_3+\mu_8)(z+x)+2yw+\mu_5(y+w)+({\mu_5}^2+2\mu_{10})$$

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we have $\xi(x,y;z,w) = \dfrac{F(x,y;z,w)dx_1dz}{(z-x)^2 f_y(x_1,y_1) f_w(z,w)}$, where

$$F(x,y;z,w) = (x^2z^3 + x^3z^2) + ({\mu_1}^2 + 2\mu_2)x^2z^2 + (\mu_3\mu_1 + \mu_4)(xz^2 + x^2z)$$
$$+ \mu_1(yz^2 + wx^2) + (2\mu_5\mu_1 + {\mu_3}^2 + 2\mu_6)xz + \mu_3(yz + wx)$$
$$+ (\mu_5\mu_3 + \mu_8)(z + x) + 2yw + \mu_5(y + w) + ({\mu_5}^2 + 2\mu_{10})$$

and

$$\eta_1 = \frac{x^2\mathbf{d}x}{f_y(x,y)}, \quad \eta_2 = \frac{(3x^3 + ({\mu_1}^2 + 2\mu_2)x^2 + (\mu_3\mu_1 + \mu_4)x + \mu_1 y)\mathbf{d}x}{f_y(x,y)}.$$

7. Periods

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$$\omega' = \left[\int_{\alpha_i} \omega_j\right],\ \omega'' = \left[\int_{\beta_i} \omega_j\right],\ \eta' = \left[\int_{\alpha_i} \eta_j\right],\ \eta'' = \left[\int_{\beta_i} \eta_j\right],$$

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and the period lattice

$$\boldsymbol{\Lambda} = \mathbb{Z}^g\, \omega' + \mathbb{Z}^g\, \omega'' \ \in\ \mathbb{C}^g.$$

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Let $\zeta_j(u)$ be the function without constant term in power series expansion w.r.t. u such that

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Namely, $\zeta_j(u) = \dfrac{\partial}{\partial u_j}\log\sigma(u)$.

It is well-known that $\sigma(u) = 0 \iff u \pmod{\Lambda} \in \Theta$, the "standard" theta divisor.

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Such function is realized by

$$\sigma(u) = c \exp\left(-\frac{1}{2}\,{}^t u \eta' {\omega'}^{-1} u\right) \vartheta\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix}({\omega'}^{-1}u \,|\, {\omega'}^{-1}\omega''),$$

where the theta series is usual one, $c = \frac{1}{D^{1/8}}\left(\frac{\det(\omega')}{(2\pi)^g}\right)^{1/2}$ with discriminant D, $\pi = 3.141592\cdots$,
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Derivatives. We denote $\sigma_{ij\ldots k}(u) = \dfrac{\partial}{\partial u_i}\dfrac{\partial}{\partial u_j} \cdots \dfrac{\partial}{\partial u_k}\sigma(u)$.

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We will define special higher derivatives

$$\sigma_{\natural^1}(u),\ \ \sigma_{\natural^2}(u),\ \ \cdots,\ \ \sigma_{\natural^{g-1}}(u),\quad (\sigma_{\natural^n}(u)=\sigma(u)\ \ \text{for}\ \ n \geq g).$$

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Note the stratification : $\{0\}=\Theta^{[0]}\subset\Theta^{[1]}\subset\cdots\subset\Theta^{[g-1]}\subset\Theta^{[g]}=J$.

Then, for $g-1>n\geq 0$ and $u\ (\mathrm{mod}\,\Lambda)\in\Theta^{[n+1]}$

$$\sigma_{\natural^{n+1}}(u)=0\iff u\ (\mathrm{mod}\,\Lambda)\in\Theta^{[n]}.$$

10. Table of $\sigma_{\natural^n}$

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(Each number in $\langle\ \rangle$ indicates $\mathbf{wt}\,(u_j)$ for $j \in \natural^n$.)

(d,p)	g	$\sharp = \natural^1$	$\flat = \natural^2$	$\natural^3$	$\natural^4$	$\natural^5$	$\natural^6$	$\natural^7$	$\natural^8$	$\cdots$
$(2,3)$	1	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,5)$	2	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,7)$	3	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,9)$	4	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,11)$	5	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,13)$	6	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(2,15)$	7	$\langle 3,7,11\rangle$	$\langle 1,5,9\rangle$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\langle 3\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(3,4)$	3	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,5)$	4	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,7)$	6	$\langle 1,6\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$(3,9)$	7	$\langle 4,10\rangle$	$\langle 2,7\rangle$	$\langle 1,5\rangle$	$\langle 4\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle\ \rangle$	$\langle\ \rangle$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

11a. Definition of $\natural^n$

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We explain by an example : $(d, q) = (3, 7)$, $g = 6$.

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					11
					8
					5
					4
					2
					1

11b. Definition of $\natural^2$ for $(d, q) = (3, 7)$, $g = 6$. (continuation)

Then, put into other boxes naturally increasing non-negative integers as follows:

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

11c. Definition of $\natural^2$ for $(d,q)=(3,7)$, $g=6$. (Continuation)

If we wish to get $\natural^n = \natural^2$, extract $(g-n)\times(g-n) = 4\times 4$ minor on the lower right corner. and Remove all rows and columns including $\mathbf{0}$.

6	7	8	9	10	11
3	4	5	6	7	8
0	1	2	3	4	5
	0	1	2	3	4
			0	1	2
				0	1

$\longrightarrow$

2	3	4	5
1	2	3	4
	0	1	2
		0	1

$\longrightarrow$

2	5
1	4

11d. Definition of $\natural^2$ for $(d,q)=(3,7)$, $g=6$. (continuation)

6	7	8	9	10	11
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0	1	2	3	4	5
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$\longrightarrow$

2	$\boxed{5}$
$\boxed{1}$	4

Finally, by reading the numbers on the off-diagonal, we have

$$\natural^2 = \langle 1,5\rangle \quad \text{and} \quad \sigma_{\natural^2}(u) = \sigma_{\langle 1,5\rangle}(u) = \frac{\partial^2}{\partial u_{\langle 1\rangle}\partial u_{\langle 5\rangle}}\sigma(u).$$

Note that $u_{\langle 1\rangle} = u_6$ and $u_{\langle 5\rangle} = u_3$ in old notation.

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Then all the points with the same x-coordinate (thers are d such points) are given by $\{(x, \gamma(y)) \mid \gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)\}$.

Since the whole space $\mathbb{C}^g$ is the pull back of $\mathbf{Sym}^g(\mathscr{C})$ with respect to mod Λ, this action of $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)$ extends to the whole space. We denote this action by

$$\gamma : u \mapsto [\gamma]u \quad \text{for } \gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1).$$

Then we see

$$\sum_{\gamma \in \mathbf{Gal}(\mathscr{C}/\mathbb{P}^1)} [\gamma]u = 0.$$

13. Properties of higher derivatives of $\sigma(u)$

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Notation (revisited):

$\mathbf{\Theta}^{[n]} = W^{[n]} \cup [-\mathbf{1}]W^{[n]}$, $W^{[n]}$ is image of $\mathbf{Sym}^n(\mathscr{C})$ via Abelian map.

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$$\sigma_{\natural^n}(u+\ell) = \chi(\ell)\, \sigma_{\natural^n}(u) \exp L(u + \tfrac{1}{2}\ell, \ell) \quad (\ell \in \Lambda).$$

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(5) $$\sigma_{\natural^{n+1}}(u+v) = \sigma_{\natural^n}(u) {v_{\langle 1\rangle}}^{w_{g-n}-g+n+1} + O({v_{\langle 1\rangle}}^{w_{g-n}+(g-n)+2})$$

for $u \bmod \Lambda \in W^{[n]}$ and $v \bmod \Lambda \in W^{[1]}$.

14. Original Frobenius-Stickelberger Formula

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Example. $y^2 + (\mu_1 x + \mu_3) y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$.

Original Frobenius-Stickelberger formula:

$$\frac{\sigma(u^{(1)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)})}{\prod_{j=1}^{n} \sigma(u^{(j)})^n} =$$

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We can give similar formula for any weightable plane curve.

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$$\frac{\sigma_{\natural^n}(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \displaystyle\prod_{i<j} \prod_{\substack{\gamma \in \mathrm{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathrm{id}}} \sigma_\flat(u^{(i)} + [\gamma]u^{(j)})}{\displaystyle\prod_{j=1}^{n} \left(\sigma_\sharp(u^{(j)})^{(d-1)(n-j)+1} \prod_{\substack{\gamma \in \mathrm{Gal}(\mathscr{C}/\mathbb{P}^1) \\ \gamma \neq \mathrm{id}}} \sigma_\sharp([\gamma]u^{(j)})^{j-1} \right)}$$

$$= \pm \left| (x^{a_j} y^{b_j})(u^{(i)}) \right|_{1 \leqq i,j \leqq n} \cdot \left| (x^{j-1})(u^{(i)}) \right|_{1 \leqq i,j \leqq n}^{d-2},$$

where $\{da_j + qb_j\}$ is the Weierstrass non-gap sequence.

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Theorem. This is OK for $(d, q) = (2,$ "any"$)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, $(5, 6)$.

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If $\boldsymbol{v} \bmod \boldsymbol{\Lambda} \in \boldsymbol{\Theta}^{[1]}$ is a variable,

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Repeating such argument, we arrive the desired properties for $\sigma_{\natural^n}(u)$.

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Exmple $(d, q) = (3, 5)$, $g = 4$; $\mathbf{Gal}(\mathscr{C}/\mathbb{P}^1) = \{\mathbf{id},\ \gamma,\ \gamma^2\}$.

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and it has poles only at $u = 0$ of order $4 \times 3 - 6 = 6$.

—. The RHS has the same zeroes and poles.

19. Proof of the first step of Frob.-Stick. Formula

Exmple $(d, q) = (3, 5)$, $g = 4$; $\mathrm{Gal}(\mathscr{C}/\mathbb{P}^1) = \{\mathrm{id},\ \gamma,\ \gamma^2\}$.

Let $v' = [\gamma]v$ and $v'' = [\gamma^2]v$

For u, $v \bmod \Lambda \in W^{[1]}$, our claim is

$$\frac{\sigma_\flat(u+v)\,\sigma_\flat(u+v')\,\sigma_\flat(u+v'')}{\sigma_\sharp(u)^3\sigma_\sharp(v)\sigma_\sharp(v')\sigma_\sharp(v'')} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}^2 \quad (\sharp = \langle 4\rangle,\ \flat = \langle 1\rangle).$$

We regard the two sides as functions of u. They are periodic with respect to Λ.

—. The LHS has zeroes at v, v', v'' of order 2 because $v + v' + v'' = 0$,

and it has poles only at $u = 0$ of order $4 \times 3 - 6 = 6$.

—. The RHS has the same zeroes and poles.

Comparing the two sides on the coefficient of leading terms of power series expansions w. r. t. $u_{\langle 1\rangle}$ show the desired formula.

Thank you very much!