Symmetry, Curves and Monopoles

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Curve results with T.P. Northover.
Monopole Results in collaboration with V.Z. Enolski, A.D'Avanzo.

Overview

Equations	*Zero Curvature/Lax* $\longrightarrow$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

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$\theta\left(t\mathbf{U} + \mathbf{C}|\tau\right)$

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at ∞_j with local coordinate $\zeta = 1/t$ $\exists$ meromorphic differential

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Ercolani-Sinha Constraints: The following are equivalent:

1. $\mathcal{L}^2$ is trivial on $\mathcal{C}$.
2. $2\mathbf{U} \in \Lambda \iff \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1} \gamma_\infty, \ldots, \oint_{\mathfrak{b}_g} \gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}.$
3. $\exists$ 1-cycle $\mathfrak{es} = \mathbf{n} \cdot \mathfrak{a} + \mathbf{m} \cdot \mathfrak{b}$ s.t. for every holomorphic differential
$$\Omega = \frac{\beta_0 \eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial \mathcal{P}}{\partial \eta}} d\zeta, \; \oint_{\mathfrak{es}} \Omega = -2\beta_0$$

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- $-K_Q = \phi_*\left(\Delta - (g-1)Q\right) = \phi_Q\left(\Delta\right)$,
  $\deg \Delta = g-1, \quad 2\Delta \equiv \mathcal{K}_{\mathcal{C}}$

Calculation

- Homology basis $\{\gamma_i\}_{i=1}^{2g} = \{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$
  - algorithm for branched covers of $\mathbb{P}^1$ (Tretkoff & Tretkoff)
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$$\oint_{\sigma_*\gamma} \omega = \oint_{\gamma} \sigma^*\omega \Longleftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} L \Longleftrightarrow M\Pi = \Pi L$$

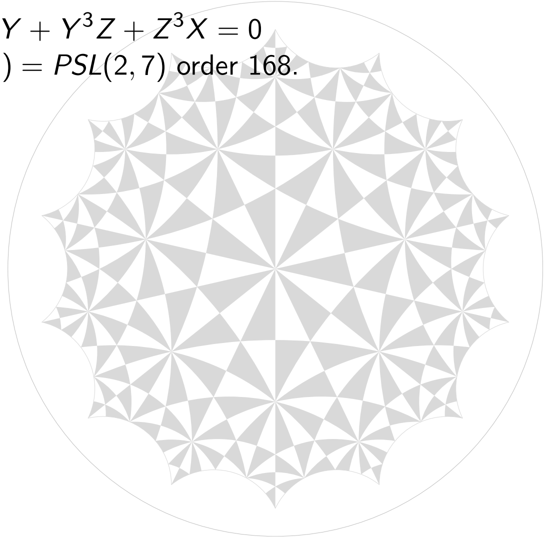
Restricts τ: $\tau B \tau + \tau A - D\tau - C = 0$

Curves with lots of symmetries: evaluate τ via character theory

Calculation

Example: Klein's Curve and Problems

- $\mathcal{C}$: $X^3Y + Y^3Z + Z^3X = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.



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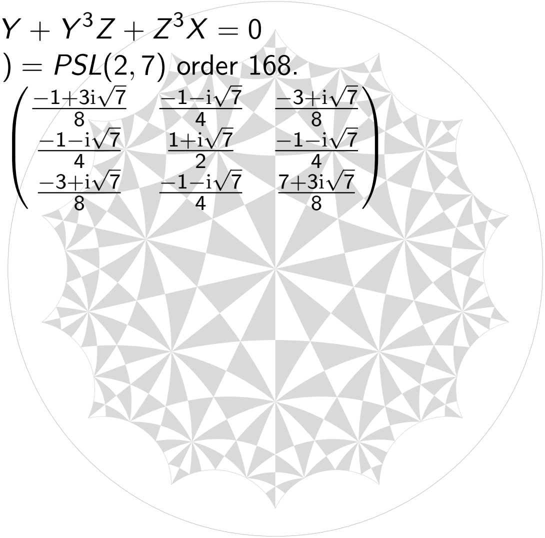

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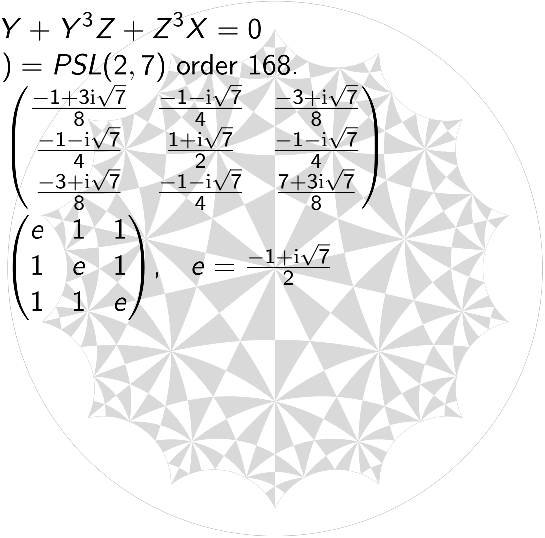

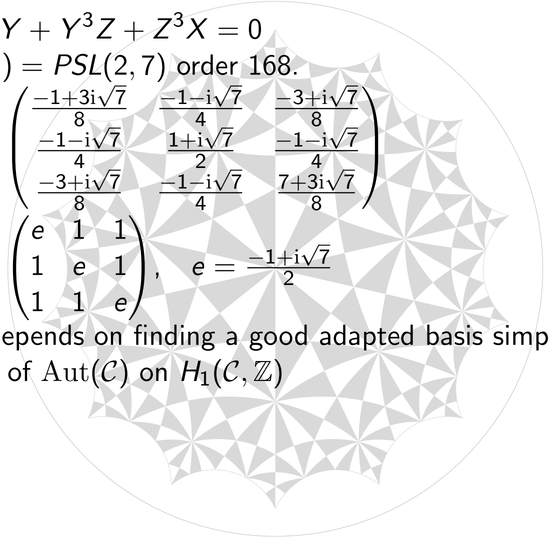

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- Symplectic Equivalence of Period Matrices τ, τ'

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z}) \Leftrightarrow M^T JM = J$$

$$\begin{pmatrix} \tau' & -1 \end{pmatrix} M \begin{pmatrix} 1 \\ \tau \end{pmatrix} = 0$$

$$\mathcal{C}:\ w^7 = (z-1)(z-\rho)^2(z-\rho^2)^4,\quad \rho = \exp(2\pi \mathrm{i}/3)$$

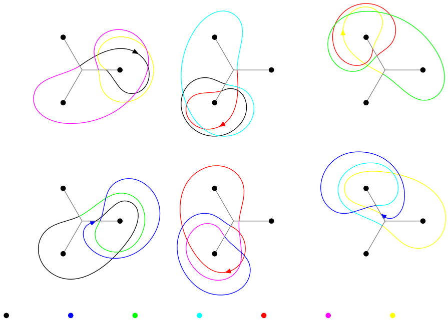

Figure: Homology basis in (z, w) coordinates

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$$-2K_Q = \phi_* \left(2\Delta - 2(g-1)Q\right) = \int_*^{2\Delta} \omega - 2(g-1)\int_*^{Q} \omega$$

$$-2K_Q.L = \int_*^{2\Delta} \sigma^*\omega - 2(g-1)\int_*^{Q} \sigma^*\omega$$

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σ^N =Id. If $L-1$ is invertible and Q a fixed point of σ then K_Q is a $2N$-torsion point.

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Corollary

Lemma+$\psi \in \mathrm{Aut}(\mathcal{C})$. Then $\int_Q^{\psi(Q)} \omega$ is a $2N(g-1)$-torsion point.

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Symmetry+Fixed point $\Rightarrow$ K_Q a torsion point.
Suppose $\exists\ l,\ m \in \mathbb{Z}^{2g}$ such that $m\Pi = l\Pi\,[L-1] = l[M-1]\Pi$.
Then $(-2K_Q + l\Pi)\,[L-1] = (n+m)\Pi$ in $\mathbb{C}$

Idea: Use Smith Normal Form of $M-1$ to choose l, $l(M-1) = m$ so as to make $n+m$ as simple as possible.

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$$M - 1 = U\,\mathrm{Diag}(d_1, \ldots, d_{2g})V, \qquad d_i | d_{i+1},\ U, V \in GL(2g, \mathbb{Z})$$

$$(mV^{-1})_i \equiv 0 \mod d_i,\ d_i > 1$$

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$$(mV^{-1})_i \equiv 0 \quad \text{mod } d_i,\ \ d_i > 1$$

Klein's curve, order 7 automorphism: $d's = 1, \ldots, 1, 7$. $Q = (0, 0)$

$$-2K_Q = (k, 0, 0, 0, 0, 0)(M-1)^{-1}\Pi, \qquad k \in \{0, 1, \ldots, 6\}$$

Symmetry+Fixed point $\Rightarrow K_Q$ a torsion point.
Suppose $\exists\ l,\ m \in \mathbb{Z}^{2g}$ such that $m\Pi = l\Pi\,[L-1] = l[M-1]\Pi$.
Then $(-2K_Q + l\Pi)\,[L-1] = (n+m)\Pi$ in $\mathbb{C}$

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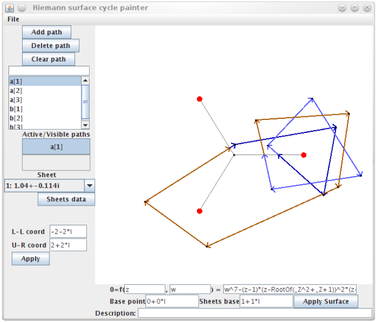
Order 4 Automorphism $\Rightarrow k = 3$. Thus $-2K_Q$ fixed. Final half-period done numerically. $K_0 = \dfrac{i}{\sqrt{7}}(3,-1,5)$

- How can one specify homology cycles?

Calculation

Techniques and Problems

- How can one specify homology cycles?



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  Example (Fay): $\phi : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$, $\phi^2 = \mathrm{Id}$, $\pi : \hat{\mathcal{C}} \to \mathcal{C} := \hat{\mathcal{C}}/ < \phi >$
  $2n$ fixed points. $\hat{g} = 2g + n - 1$

$$\mathfrak{a}_1, \mathfrak{b}_1, \ldots \mathfrak{a}_g, \mathfrak{b}_g, \mathfrak{a}_{g+1}, \mathfrak{b}_{g+1}, \ldots \mathfrak{a}_{g+n+1}, \mathfrak{b}_{g+n+1}, \mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$$

where $\mathfrak{a}_{1'}, \mathfrak{b}_{1'}, \ldots, \mathfrak{a}_{g'}, \mathfrak{b}_{g'}$ a basis of $H_1(\mathcal{C}, \mathbb{Z})$ and

$$\mathfrak{a}_{\alpha'} + \phi(\mathfrak{a}_\alpha) = 0 = \mathfrak{b}_{\alpha'} + \phi(\mathfrak{b}_\alpha), \qquad 1 \leq \alpha \leq g$$
$$\mathfrak{a}_i + \phi(\mathfrak{a}_i) = 0 = \mathfrak{b}_i + \phi(\mathfrak{b}_i), \qquad g + 1 \leq i \leq g + n - 1$$

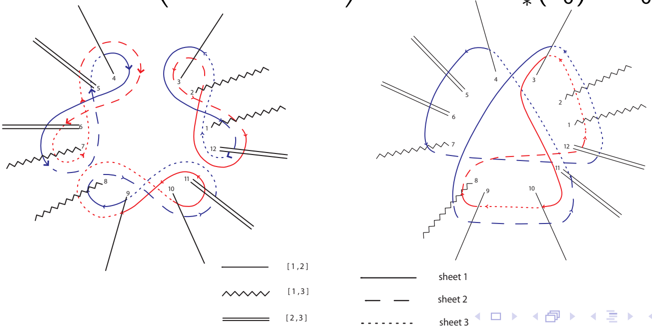
Calculation: The spectral curve of genus 4

$$\hat{\mathcal{C}}: \qquad w^3 + \alpha w z^2 + \beta z^6 + \gamma z^3 - \beta = 0$$

$$\mathrm{C}_3 : (z, w) \mapsto (\rho z, \rho w), \;\; \rho = \exp(2\pi \mathrm{i}/3)$$

$$\tau_{\hat{\mathcal{C}}_{\mathrm{monopole}}} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\begin{aligned} \sigma_*^k(\mathfrak{a}_i) &= \mathfrak{a}_{i+k} \\ \sigma_*^k(\mathfrak{b}_i) &= \mathfrak{b}_{i+k} \\ \sigma_*^k(\mathfrak{a}_0) &= \mathfrak{a}_0 \\ \sigma_*^k(\mathfrak{b}_0) &\sim \mathfrak{b}_0 \end{aligned}$$



Calculation

The spectral curve of genus 2

$$\mathcal{C} = \hat{\mathcal{C}}/C_3 : \qquad y^2 = (x^3 + \alpha x + \gamma)^2 + 4\beta^2$$

$$\tau = \begin{pmatrix} \frac{a}{3} & b \\ b & c + 2d \end{pmatrix}$$

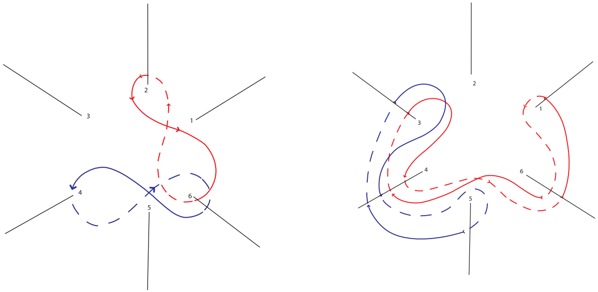


Figure: Projection of the previous basis

Cyclically Symmetric Monopoles

- $\omega = \exp(2\pi i/n)$, $(\eta, \zeta) \to (\omega\eta, \omega\zeta)$
  $\mathtt{C}_n$ symmetric (centred) charge-n monopole curve of form

  $$\hat{\mathcal{C}}: \quad \eta^n + a_2\eta^{n-2}\zeta^2 + \ldots + a_n\zeta^n + \beta\zeta^{2n} + (-1)^n\beta = 0, \quad a_i, \beta \in \mathbf{R}$$

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  $$\mathcal{C}: \quad y^2 = (x^n + a_2x^{n-2} + \ldots + a_n)^2 - 4(-1)^n\beta^2$$

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Theorem

Any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations.

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- Fay-Accola

$$\theta[\mathbf{C}](\pi^* z;\, \tau_{\mathrm{monopole}}) = c \prod_{i=1}^{n} \theta[\mathbf{e}_i](z;\, \tau_{\mathrm{Toda}})$$

"*θ-functions are still far from being a spectator sport.*"(L.V. Ahlfors)

C_3 Cyclically Symmetric Monopoles

- $\mathfrak{c} := \pi(\mathfrak{es})$
  $Y^2 = (X^3 + a\,X + g)^2 + 4$

$$\text{ES conditions} \equiv \oint_{\mathfrak{c}} \frac{\mathrm{d}X}{Y} = 0$$

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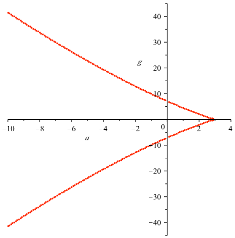

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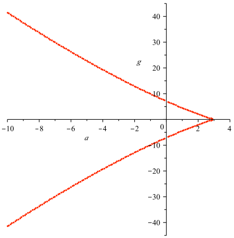

- With $a = \alpha/\beta^{2/3}$, $g = \gamma/\beta$ and β defined by

  $$6\beta^{1/3} = \oint_{\mathfrak{c}} \frac{X\mathrm{d}X}{Y}$$

  we may recover the monopole spectral curve.

C_3 Cyclically Symmetric Monopoles

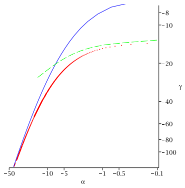

Figure: A log-log plot of the asymptotic behaviour of α versus γ