Monopoles and monopole-antimonopole systems

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FAKULTÄT V

The higher-genus sigma function and applications
Edinburgh, Oct 12 2010


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Outline

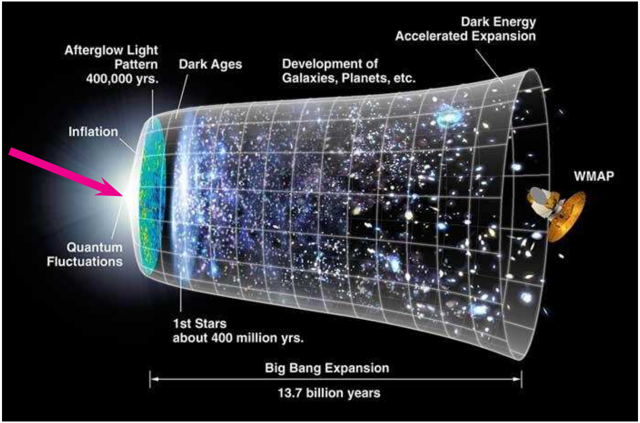
1. Monopoles in Flat Space
   - Non-Abelian Monopoles
   - Multimonopoles
   - Monopole-Antimonopole Pairs
   - Monopole-Antimonopole Systems
   - Dyons and Rotation
2. Gravitating Monopoles and Black Holes
   - Monopoles in Curved Space
   - Black Holes within Monopoles

Outline

Yang-Mills-Higgs Monopoles

't Hooft 1974,
Polyakov 1974

- globally regular static solutions
- finite energy
- magnetic charge
- gauge group SU(2)



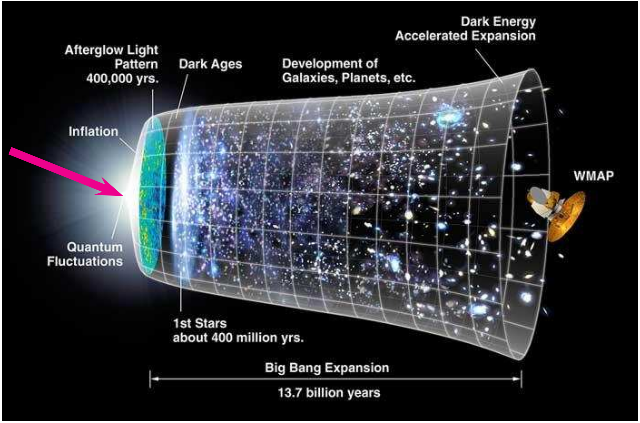
Yang-Mills-Higgs Monopoles

GUTs
generic prediction
magnetic monopoles

$t \approx 10^{-34}$ s

$T \approx 10^{14}$ GeV

- at GUT phase transition
- monopoles: huge mass
- motivation for inflation



SU(2) Yang-Mills-Higgs Theory

Lagrangian $\mathcal{L}$

$$\mathcal{L} = -\underbrace{\frac{1}{2}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}}_{\text{gauge field}} - \underbrace{\frac{1}{4}\mathrm{Tr}\left\{D_\mu\Phi D^\mu\Phi\right\}}_{\text{Higgs field}} - \underbrace{\frac{\lambda}{8}\mathrm{Tr}\left(\Phi^2-\eta^2\right)^2}_{\text{Higgspotential}}$$

gauge field $\quad A_\mu = A_\mu^a \tau_a$

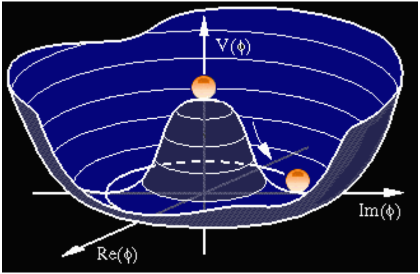
field strength tensor $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

Higgs field triplet $\quad \Phi = \Phi^a \tau_a$

covariant derivative $\quad D_\mu\Phi = \partial_\mu\Phi + ig[A_\mu, \Phi]$

constants $\quad g, \lambda, \eta$

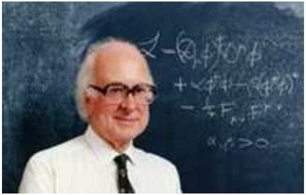
SU(2) Yang-Mills-Higgs Theory



Higgs Potential $V(\Phi)$
"Mexican hat"–Potential

$$V(\Phi) = \frac{\lambda}{8} \mathrm{Tr} \left(\Phi^2 - \eta^2\right)^2$$

Peter
Higgs
1929



spontaneous symmetry breaking
SU(2) $\longrightarrow$ U(1)

gauge bosons: $m_{W^\pm} = g\eta \ , \quad m_\gamma = 0$

Higgs boson: $m_H = \sqrt{2\lambda}\eta$

Monopoles: Static Solutions with Finite Energy

energy functional

$$E = \int d^3x \left[\frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2} D_i \Phi^a D^i \Phi^a + \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2 \right]$$

static equations of motion

$$D_i F^{aij} = g\epsilon^{abc} (D^j \Phi^b) \Phi^c$$

$$D_i D^i \Phi^a = -\lambda (\Phi^b \Phi^b) \Phi^a + \lambda \eta^2 \Phi^a$$

finite energy: boundary conditions

$$|\Phi| \stackrel{r\to\infty}{\longrightarrow} \eta \quad \Longleftrightarrow \quad \Phi^a/\eta \stackrel{r\to\infty}{\longrightarrow} \hat{\Phi}^a_\infty \ \text{ unit vector}$$

Higgs field at infinity induces mapping from physical space to internal space

$$\hat{\Phi}_\infty(\theta,\varphi) \ : \quad S^2_\infty(\theta,\varphi) \longrightarrow S^2 \ : \quad \pi_2(S^2) = Z$$

Monopoles: Static Solutions with Finite Energy

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Higgs field at infinity induces mapping from physical space to internal space

$$\hat{\Phi}_\infty(\theta, \varphi) \quad : \quad S^2_\infty(\theta, \varphi) \longrightarrow S^2 \quad : \quad \pi_2(S^2) = Z$$

degree of map

$$\boxed{n = \frac{-i}{8\pi} \int_{S^2_\infty} \mathrm{Tr}\{\hat{\Phi} \partial_\theta \hat{\Phi} \partial_\varphi \hat{\Phi}\} d\theta d\varphi}$$

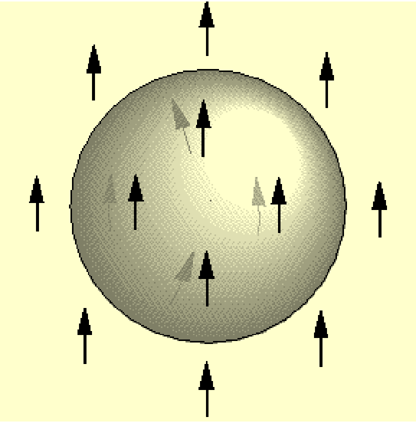
energy bound

$$\boxed{E_n \geq 4\pi\eta \frac{|n|}{g}}$$

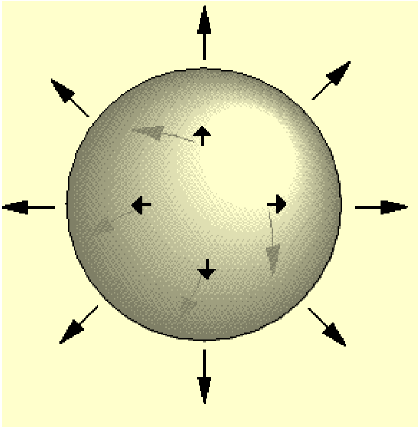
n integer

Mappings

$n = 0$ mapping:
vacuum configuration $\vec{\Phi} = \eta \vec{e}_z$



$n = 1$ mapping:
hedgehog configuration $\vec{\Phi} = \eta \vec{e}_r$



Magnetic Charge

gauge invariant electromagnetic field strength tensor

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \mathrm{Tr}\left\{\hat{\Phi}F_{\mu\nu} - \frac{i}{2g}\hat{\Phi}D_\mu\hat{\Phi}D_\nu\hat{\Phi}\right\} \\ &= \mathrm{Tr}\left\{\partial_\mu(\hat{\Phi}A_\nu) - \partial_\nu(\hat{\Phi}A_\mu)\right\} - \frac{i}{2g}\mathrm{Tr}\left\{\hat{\Phi}\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi}\right\} \end{aligned}$$

when $\hat{\Phi} = \vec{e}_z \cdot \vec{\tau}$

$$\mathcal{F}_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$$

Magnetic Charge

gauge invariant electromagnetic field strength tensor

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \mathrm{Tr}\left\{\hat{\Phi}F_{\mu\nu} - \frac{i}{2g}\hat{\Phi}D_\mu\hat{\Phi}D_\nu\hat{\Phi}\right\} \\ &= \mathrm{Tr}\left\{\partial_\mu(\hat{\Phi}A_\nu) - \partial_\nu(\hat{\Phi}A_\mu)\right\} - \frac{i}{2g}\mathrm{Tr}\left\{\hat{\Phi}\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi}\right\}\end{aligned}$$

magnetic field has non-zero divergence

$$\vec{\nabla}\cdot\vec{\mathcal{B}} = \frac{4\pi}{g}k^0$$

with topological current k^μ

$$k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c$$

Magnetic Charge

gauge invariant electromagnetic field strength tensor

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= \mathrm{Tr}\left\{\hat{\Phi}F_{\mu\nu} - \frac{i}{2g}\hat{\Phi}D_\mu\hat{\Phi}D_\nu\hat{\Phi}\right\} \\
&= \mathrm{Tr}\left\{\partial_\mu(\hat{\Phi}A_\nu) - \partial_\nu(\hat{\Phi}A_\mu)\right\} - \frac{i}{2g}\mathrm{Tr}\left\{\hat{\Phi}\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi}\right\}
\end{aligned}$$

magnetic field has non-zero divergence

$$\vec{\nabla}\cdot\vec{\mathcal{B}} = \frac{4\pi}{g}k^0$$

magnetic charge

$$\boxed{P = \frac{1}{4\pi}\int_{S^2_\infty}\mathcal{F}_{\theta\varphi}d\theta d\varphi = \frac{n}{g}}$$

Spherically Symmetric Monopoles: $n = 1$

Wu-Yang Ansatz

$$A_\mu dx^\mu = \frac{1}{2g}\left[\tau_\varphi\left(1 - K(r)\right)d\theta - \tau_\theta\left(1 - K(r)\right)\sin\theta d\varphi\right]$$

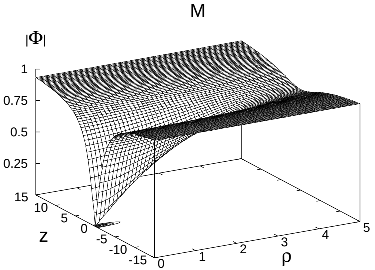
$$\Phi = \eta H(r)\tau_r \qquad (\tau_r = \vec{\tau}\cdot\vec{e}_r\ , \quad \tau_\theta = \vec{\tau}\cdot\vec{e}_\theta\ , \quad \tau_\varphi = \vec{\tau}\cdot\vec{e}_\varphi)$$

boundary conditions

$$K(0) = 1\ , \quad K(\infty) = 0$$
$$H(0) = 0\ , \quad H(\infty) = 1$$

monopole properties

- magnetic charge: $P = \dfrac{1}{g}$



Spherically Symmetric Monopoles: $n = 1$

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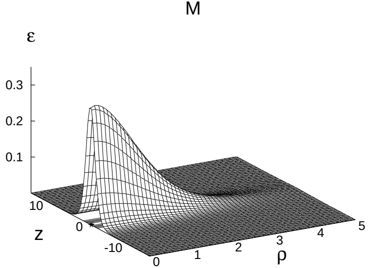
monopole properties

- magnetic charge: $P = \dfrac{1}{g}$
- size: core $\approx$ $(m_W)^{-1} = (g\eta)^{-1}$



Spherically Symmetric Monopoles: $n = 1$

Wu-Yang Ansatz

$$A_\mu dx^\mu = \frac{1}{2g}\left[\tau_\varphi \left(1 - K(r)\right) d\theta - \tau_\theta \left(1 - K(r)\right) \sin\theta d\varphi\right]$$

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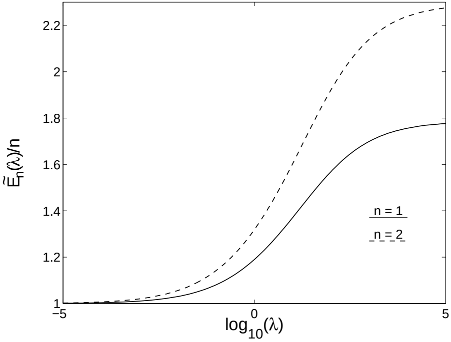
monopole properties

- magnetic charge: $P = \dfrac{1}{g}$
- monopole mass:
  $E = \dfrac{4\pi\eta}{g} f(\lambda) \geq \dfrac{4\pi\eta}{g}$



Bogomol'nyi Bound

Bogomol'nyi 1976

energy functional

$$E = \int d^3x \left[\frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2} D_i \Phi^a D^i \Phi^a + \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2 \right]$$

first and second term

$$\int d^3x \left[\frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2} D_i \Phi^a D^i \Phi^a \right]$$
$$= \int d^3x \sum_{ija} \frac{1}{4} \left(F^a_{ij} - \varepsilon_{ijk} D_k \Phi^a \right)^2 + \int d^3x \frac{1}{2} \varepsilon_{ijk} F^a_{ij} D_k \Phi^a$$

energy functional

$$E = \int d^3x \sum_{ija} \frac{1}{4} \left(F^a_{ij} - \varepsilon_{ijk} D_k \Phi^a \right)^2 + \frac{4\pi n \eta}{g} + \int d^3x \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2$$

Bogomol'nyi Bound

$$E \geq \frac{4\pi n \eta}{g}$$

BPS Limit

Bogomol'nyi 1976, Prasad, Sommerfield 1975

vanishing Higgs self-coupling: $\lambda = 0$

Bogomol'nyi equations (1st order)

$$F^a_{ij} = \epsilon_{ijk} D_k \Phi^a$$

energy saturates Bogomol'nyi bound

$$E_{\rm BPS} = n \, \frac{4\pi\eta}{g}$$

exact solutions: $n = 1$

$$K_{\rm BPS}(r) = \frac{g\eta r}{\sinh(g\eta r)}$$

$$H_{\rm BPS}(r) = \frac{1}{\tanh(g\eta r)} - \frac{1}{g\eta r}$$

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN)

Nahm equation

$$\frac{dT_i(s)}{ds} = \frac{1}{2}\varepsilon_{ijk}\left[T_j(s), T_k(s)\right]$$

$n \times n$ Nahm matrices $T_1(s)$, $T_2(s)$, $T_3(s)$ defined on $-1 \leq s \leq 1$

Weyl equation

$$\left(1_{2n}\frac{d}{ds} + iT_j(s) \otimes \tau_j - 1_n \otimes x^j \tau_j\right) v(s) = 0$$

two normalizable spinors $v_a(s) : a = 1, 2$

$$\int_{-1}^{1} v_a^\dagger(s) v_b(s) ds = \delta_{ab}$$

Higgs field

$$\Phi(\vec{x})_{ab} = i\int_{-1}^{1} s v_a^\dagger(s) v_b(s) ds$$

gauge field

$$A_i(\vec{x})_{ab} = \int_{-1}^{1} s v_a^\dagger(s) \frac{\partial}{\partial x^i} v_b(s) ds$$

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN): $n = 1$

charge-1 monopole

1×1 Nahm matrices: $T_i(s) = ic_i$, c_i constant

Weyl equation

$$\left(\frac{d}{ds} - \tau \cdot \vec{x}\right) v(s) = 0$$

two solutions

$$v_1(s) = \sqrt{\frac{r}{2\sinh r}} \left(\cosh \frac{sr}{2} + \sinh \frac{sr}{2}\, \hat{x} \cdot \tau\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2(s) = \sqrt{\frac{r}{2\sinh r}} \left(\cosh \frac{sr}{2} + \sinh \frac{sr}{2}\, \hat{x} \cdot \tau\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Higgs field

$$\Phi(\vec{x})_{ab} = i \int_{-1}^{1} s v_a^\dagger(s) v_b(s) ds$$

$$\Phi(\vec{x})_{ab} = i \left(\coth r - \frac{1}{r}\right) \hat{x} \cdot \tau_{ab}$$

Outline

1. Monopoles in Flat Space
   - Non-Abelian Monopoles
   - Multimonopoles
   - Monopole-Antimonopole Pairs
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Axially Symmetric Multimonopoles

Rebbi, Rossi 1980

$n > 1$: axially symmetric multimonopoles

$$A_\mu dx^\mu = \frac{1}{2gr}\left[\tau^n_\phi \left(H_1 dr + (1-H_2)\, r d\theta\right) - n\left(\tau^n_r H_3 + \tau^n_\theta (1-H_4)\right) r \sin\theta d\phi\right]$$

$$\Phi = \Phi_1 \tau^n_r + \Phi_2 \tau^n_\theta$$

$$\begin{aligned} \tau^n_r &= \vec{\tau}\cdot(\sin\theta\cos n\varphi, \sin\theta\sin n\varphi, \cos\theta) \\ \tau^n_\theta &= \vec{\tau}\cdot(\cos\theta\cos n\varphi, \cos\theta\sin n\varphi, -\sin\theta) \\ \tau^n_\varphi &= \vec{\tau}\cdot(-\sin n\varphi, \cos n\varphi, 0) \end{aligned}$$

winding number $n \Longrightarrow$ magnetic charge $P = \frac{n}{g}$

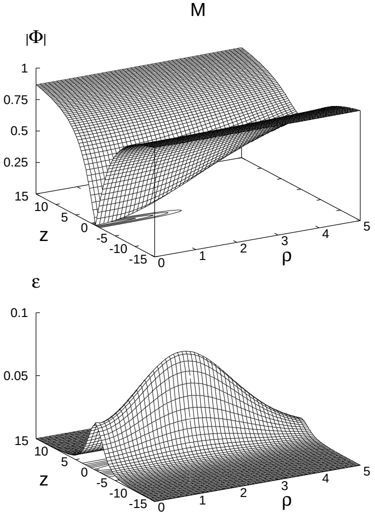
Axially Symmetric Multimonopoles

Ward 1981
Forgacs, Horvath, Palla 1981
Prasad, Rossi 1981
Corrigan, Goddard 1981

exact BPS multimonopoles

properties

- n superimposed monopoles located at origin
- Higgs field zero $|\Phi| = 0$ at origin
- energy density: torus



Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 2$

general charge-2 monopoles

Nahm equation

$$\frac{dT_i(s)}{ds} = \frac{1}{2}\varepsilon_{ijk}\left[T_j(s), T_k(s)\right]$$

2×2 Nahm matrices $T_1(s)$, $T_2(s)$, $T_3(s)$

$$T_1(s) = \frac{i}{2} f_1(s)\tau_1 \ , \quad T_2(s) = \frac{i}{2} f_2(s)\tau_2 \ , \quad T_3(s) = -\frac{i}{2} f_3(s)\tau_3$$

f_i satisfy the Euler equations

$$\frac{df_1}{ds} = f_2 f_3 \ , \quad \frac{df_2}{ds} = f_3 f_1 \ , \quad \frac{df_3}{ds} = f_1 f_2$$

scaling symmetry $\quad f_j(s) = L F_j(u), \quad u = L(s + s_0)$

L, s_0 arbitrary constants

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 2$

general charge-2 monopoles

functions $f_i(s)$: elliptic functions

$$f_1 = \frac{-L\mathrm{dn}_k(u)}{\mathrm{sn}_k(u)} \ , \quad f_2 = \frac{-L}{\mathrm{sn}_k(u)} \ , \quad f_3 = \frac{-L\mathrm{cn}_k(u)}{\mathrm{sn}_k(u)}$$

integration constant k

$\mathrm{sn}_k(u)$ has zeros at $u = 0$ and $u = 2K_k$

$$K_k = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

complete elliptic integral of the first kind

functions F_i have required poles at $s = \pm 1$ when $L = K_k$, $s_0 = 1$

result: 1-parameter family of Nahm data parametrized by k

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 2$

general charge-2 monopoles

Weyl equation

$$\left(1_{2n}\frac{d}{ds} + iT_j(s)\otimes\tau_j - 1_n\otimes x^j\tau_j\right)v(s) = 0$$

two normalizable spinors $v_a(s) : a = 1, 2$

$$\int_{-1}^{1} v_a^\dagger(s)v_b(s)ds = \delta_{ab}$$

Brown, Panagopoulos, Prasad 1982
2 separated monopoles in the ADHMN construction

analytical solutions for v only on the axis connecting the 2 monopoles

analytical expression for Higgs field only on the axis connecting the 2 monopoles

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 2$

general charge-2 monopoles

Weyl equation

$$\left(1_{2n}\frac{d}{ds} + iT_j(s)\otimes\tau_j - 1_n\otimes x^j\tau_j\right)v(s) = 0$$

Manton and Sutcliffe, 2004
numerical solutions for $v(s)$

$k = 0.99$ $k = 0.7$ $k = 0$

parameter k:
measure of the splitting of the $n = 2$ monopole into 2 $n = 1$ monopoles

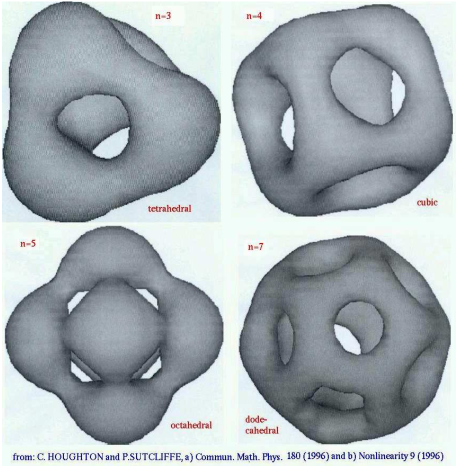
Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n > 2$

charge-n monopoles

Nahm data are known

- $n > 2$ axial
- $n = 3$ tetrahedral
- $n = 4$ cubic
- $n = 5$ octahedral
- $n = 7$ dodecahedral
- ...

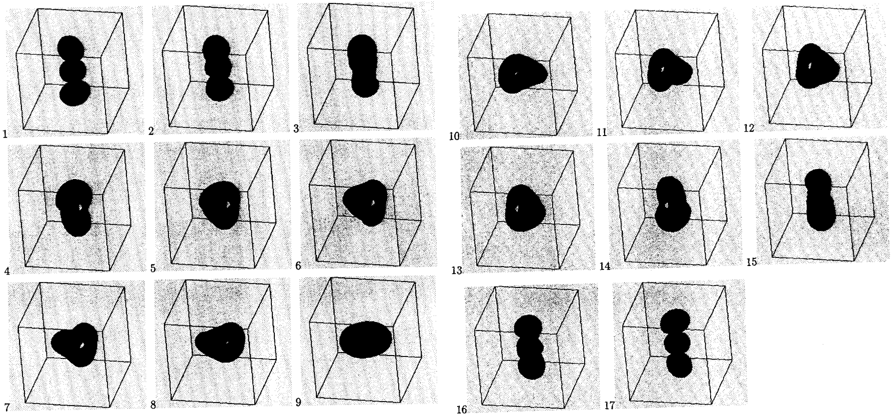
numerical solution
of Weyl equation



Houghton, Sutcliffe 1996

Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 3$

Houghton, Sutcliffe 1996



spectral curve $\eta^3 - 6(a^2 + 4\epsilon)^{1/3}\kappa^2\eta\zeta^2 + 2i\kappa^3 a(\zeta^5 - \zeta) = 0$

$\epsilon = \pm 1$, $a \in R$, 2κ period of elliptic curve $y^2 = 4x^3 - 3(a^2 + 4\epsilon)^{2/3}x + 4\epsilon$

Multimonopoles and Rational Maps

Donaldson 1984

classification of multimonopoles with rational maps between spheres

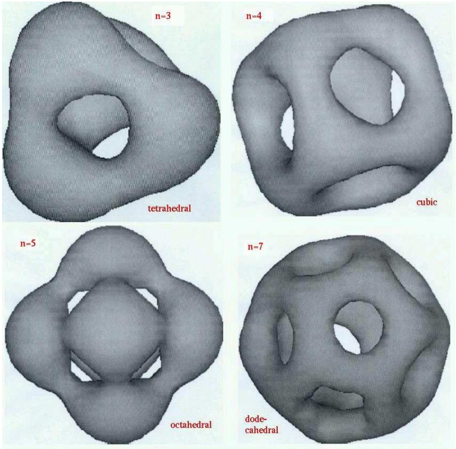
$$\mathcal{R}(z) = \frac{p(z)}{q(z)} \ , \quad z = \tan\frac{\theta}{2}e^{i\varphi}$$

point z on the sphere corresponds to unit vector $\vec{n}_z$

$$\frac{1}{1+|z|^2}\left(2\mathrm{Re}\, z, 2\mathrm{Im}\, z, 1-|z|^2\right)$$

value of the rational map $\mathcal{R}$ is associated with unit vector $\vec{n}_\mathcal{R}$

$$\frac{1}{1+|\mathcal{R}|^2}\left(2\mathrm{Re}\,\mathcal{R}, 2\mathrm{Im}\,\mathcal{R}, 1-|\mathcal{R}|^2\right)$$



from: C. HOUGHTON and P.SUTCLIFFE, a) Commun. Math. Phys. 180 (1996) and b) Nonlinearity 9 (1996)

Houghton, Sutcliffe 1996

Multimonopoles and Rational Maps

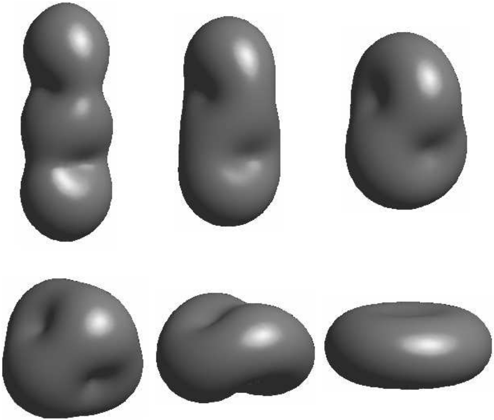
Donaldson 1984
classification of multimonopoles with rational maps between spheres

Houghton, Sutcliffe 1996
example: $n = 3$

$$\mathcal{R}(z) = \frac{\sqrt{3}az^2 - 1}{z(z^2 - \sqrt{3}a)}$$

$a = 0$: torus

$a = 1$: tetrahedron



$\lambda \neq 0$: Kleihaus, Kunz 2003

Multimonopoles and Rational Maps

Donaldson 1984
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Houghton, Sutcliffe 1996
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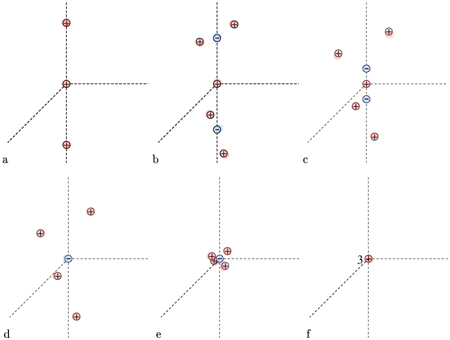
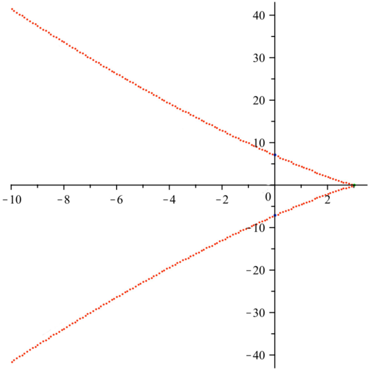


Atiyah-Drinfeld-Manin-Hitchin-Nahm (ADMHN) $n = 3$

Braden, Enolski, 2010



spectral curve $\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0$

Outline

Monopole-Antimonopole Pairs

Taubes 1982

BPS-limit: in each topological sector exist an infinite number of solutions with

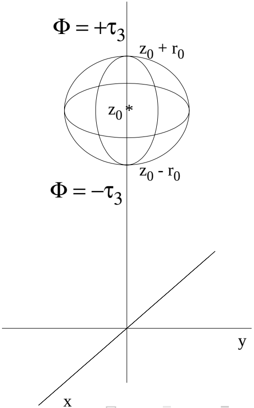
$$E > 4\pi\eta \frac{|n|}{g}$$

Nahm, Rüber 1985

vacuum sector $n = 0$

$$\Phi = 0 \text{ at } z_0 \quad \Leftrightarrow \quad \Phi = 0 \text{ at } -z_0$$

monopole at z_0 $\Leftrightarrow$ antimonopole at z_0



sphere S^2 of radius r_0 centered at z_0

$$n = -\frac{i}{8\pi r_0^2} \int_{S_2} \mathrm{Tr}(\hat{\Phi} d\hat{\Phi} \wedge d\hat{\Phi}) = \pm 1$$

magnetic dipole field

$$A_i^3 dx^i \to \frac{C_\mathrm{m}}{r^3} (\vec{e}_z \times \vec{r})_i dx^i$$

Monopole-Antimonopole Pairs

Nahm, Rüber 1985
Kleihaus, Kunz 2000

$$A_\mu dx^\mu = \frac{1}{2gr}\left[\tau_\varphi \left(H_1 dr + 2\left(1-H_2\right) r d\theta\right) -2\left(\bar\tau_r^2 H_3 + \bar\tau_\theta^2 \left(1-H_4\right)\right) r \sin\theta d\varphi\right]$$

$$\Phi = \Phi_1(r,\theta)\bar\tau_r^{(2)} + \Phi_2(r,\theta)\bar\tau_\theta^{(2)}$$

$$\tau_\rho = \cos\varphi\tau_1 + \sin\varphi\tau_2 \ , \qquad \tau_\varphi = -\sin\varphi\tau_1 + \cos\varphi\tau_2$$

$$\bar\tau_r^2 = \sin 2\theta\tau_\rho + \cos 2\theta\tau_3 \ , \qquad \bar\tau_\theta^2 = \cos 2\theta\tau_\rho - \sin 2\theta\tau_3$$

boundary conditions ($r \longrightarrow \infty$)

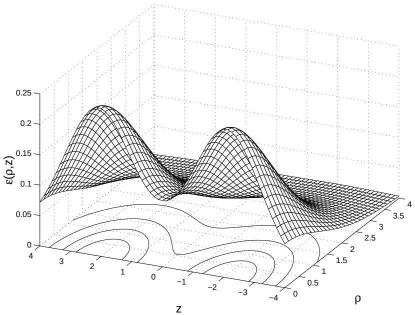
$$H_1 \to 0 \ , \quad H_2 \to 0 \ , \quad H_3 \to \sin\theta \ , \quad 1 - H_4 \to \cos\theta$$
$$\Phi_1 \to \eta \ , \quad \Phi_2 \to 0$$

no net magnetic charge

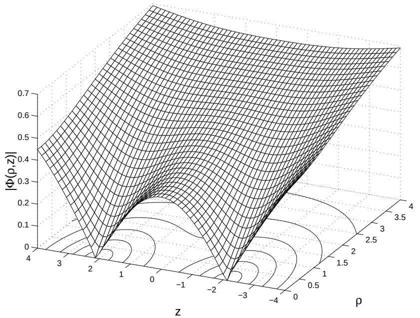
Monopole-Antimonopole Pairs

Kleihaus, Kunz 2000

monopol-antimonopole pairs: MAPs



energy density



Higgs field modulus

Monopole-Antimonopole Pairs

Kleihaus, Kunz 2000

monopol-antimonopole pairs: MAPs

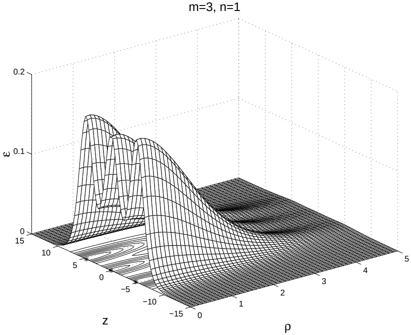
λ	$E\frac{g}{4\pi\eta}$	$E_\infty\frac{g}{4\pi\eta}$	d	$\lvert\Phi(0)\rvert$	C_m
0	1.697	2.000	4.23	0.328	2.36
0.01	2.015	2.204	3.34	0.489	1.84
0.1	2.330	2.498	3.26	0.791	1.71
1.0	2.713	2.900	3.0	0.986	1.57
10.0	3.042	3.241	3.0	0.9996	1.55

Outline

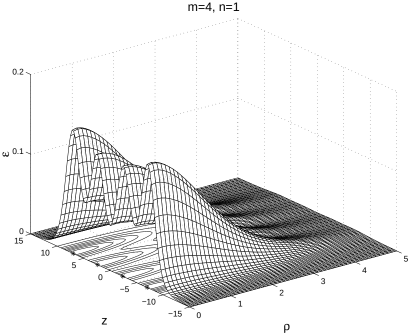
Monopole-Antimonopole Chains

Kleihaus, Kunz, Shnir 2003

monopol-antimonopole chains: MACs



energy density: $m = 3$

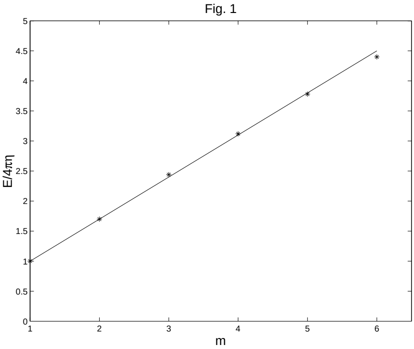


energy density: $m = 4$

Monopole-Antimonopole Chains

Kleihaus, Kunz, Shnir 2003

monopol-antimonopole chains: MACs



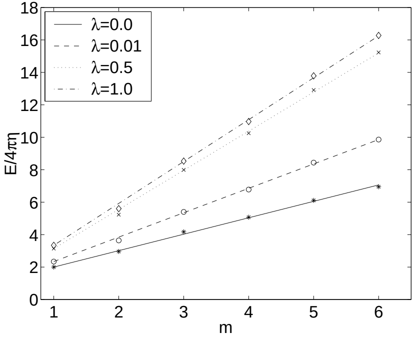
energy versus number of (anti)monopoles

Monopole-Antimonopole Chains

Kleihaus, Kunz, Shnir 2003

monopol-antimonopole chains: MACs

Fig. 14

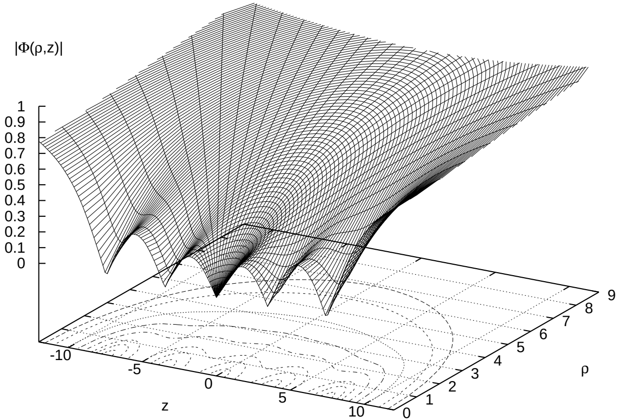


energy versus number of (anti)monopoles

Monopole-Antimonopole Chains

Kleihaus, Kunz, Shnir 2003

monopol-antimonopole chains: MACs

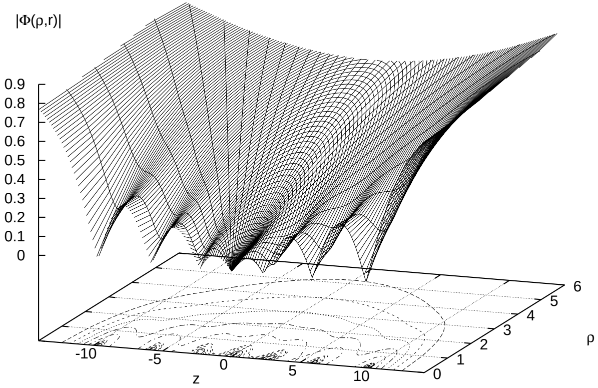


MACs

Monopole-Antimonopole Chains

Kleihaus, Kunz, Shnir 2003

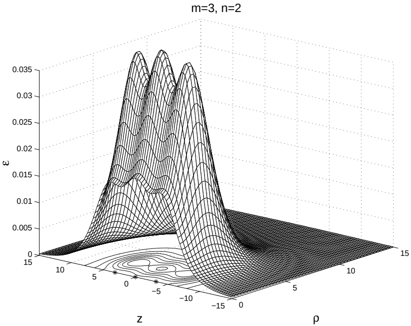
monopol-antimonopole chains: MACs



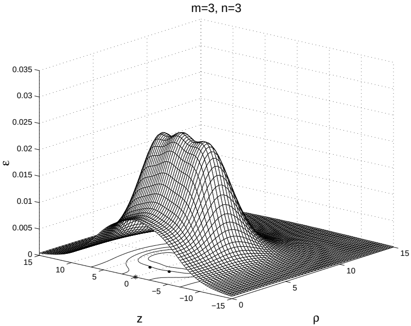
Big MACs

Monopole-Antimonopole Chains with Higher Charge?

Kleihaus, Kunz, Shnir 2003



energy density: $m = 3$, $n = 2$

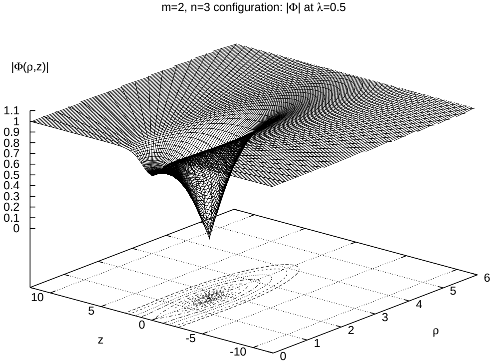


energy density: $m = 3$,$m = 3$

Monopole-Antimonopole Vortex Rings

Kleihaus, Kunz, Shnir 2003

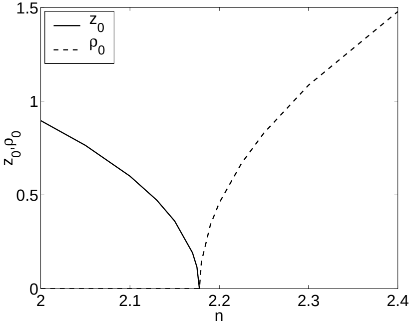
monopol-antimonopole vortex rings



Higgs field modulus: $m = 2$, $n = 4$, $\lambda = 0.5$

Monopole-Antimonopole Vortex Rings

Kleihaus, Kunz, Shnir 2003
monopol-antimonopole vortex rings

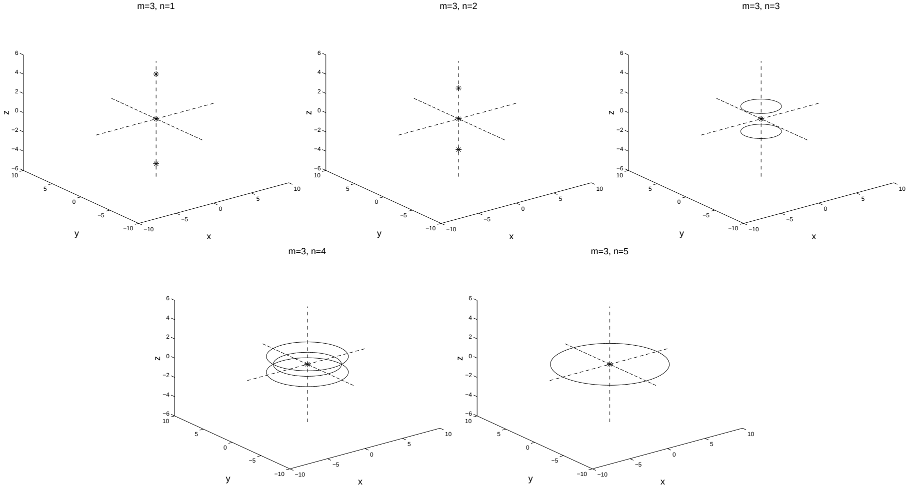


evolution of the nodes of the Higgs field: $m = 2$, n

Monopole-Antimonopole Vortex Rings

Kleihaus, Kunz, Shnir 2003

monopol-antimonopole vortex rings



evolution of the nodes of the Higgs field: $m = 3$, n

Outline

Electrically Charged Solutions

Julia and Zee, 1975

BPS limit

solution of the field equations with $||\tilde{\Phi}^a|| \overset{r\to\infty}{\longrightarrow} \tilde{\eta}$

$$\left(A_i^a \ , \quad A_0^a = 0 \ , \quad \tilde{\Phi}^a\right)$$

solution with $||\Phi^a|| \overset{r\to\infty}{\longrightarrow} \tilde{\eta}\sqrt{1+Q^2} = \eta$ and electric charge Q

$$\left(A_i^a \ , \quad A_0^a = Q\tilde{\Phi}^a \ , \quad \Phi^a = \sqrt{1+Q^2}\tilde{\Phi}^a\right)$$

- monopoles $\longrightarrow$ dyons

$$E(Q) = \frac{4\pi|n|}{g}\eta\sqrt{1+Q^2}$$

- monopole-antimonopole systems $\longrightarrow$ electrically charged systems

Electrically Charged Rotating Solutions?

van der Bij and Radu, 2001

axially symmetric solutions
generically angular momentum density T_φ^t

$$J = \int T_\varphi^t \sqrt{-g} d^3x = \int 2\mathrm{Tr}\{F_{r\varphi}F^{rt} + F_{\theta\varphi}F^{\theta t} + D_\varphi \Phi D^t \Phi\}\sqrt{-g} d^3x$$

angular momentum J?

- solutions with magnetic charge $(P \neq 0)$

$$J = 0$$

- solutions without magnetic charge $(P = 0)$

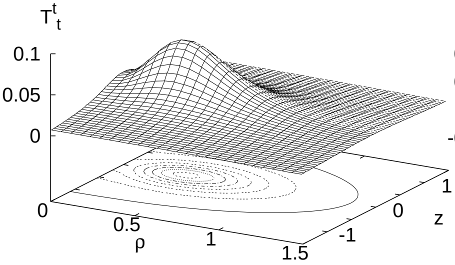
$$J \sim Q$$

$$J \sim nQ(1-\sigma)\ , \quad P = n\sigma\ , \quad \sigma = \frac{1}{2}\left[1-(-1)^m\right]$$

Electrically Charged Rotating Solutions?

Kleihaus, Kunz, Neemann 2005

multimonopoles: $P \neq 0 \Longrightarrow J = 0$

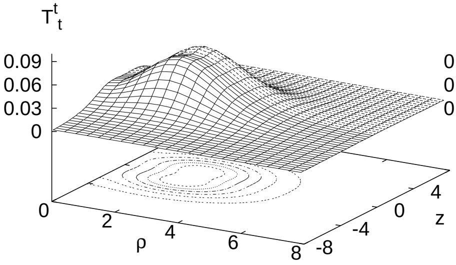


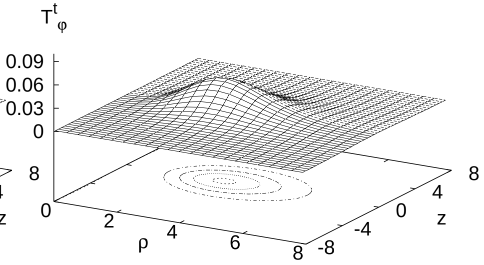
energy density: $m = 1,\, n = 2$ angular momentum density: $m = 1,\, n = 2$

Electrically Charged Rotating Solutions?

Kleihaus, Kunz, Neemann 2005

monopole-antimonopole pairs: $P = 0 \Longrightarrow J \neq 0$





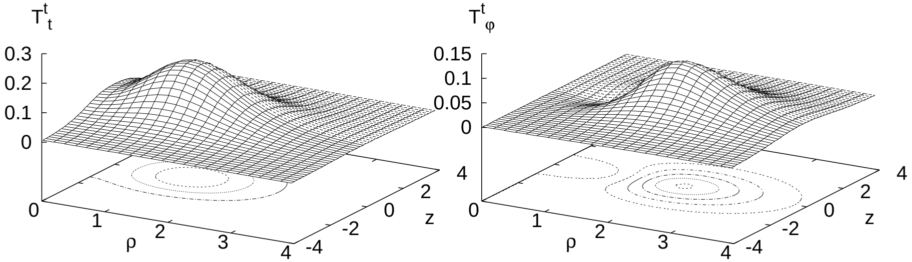
energy density: $m = 2$, $n = 2$

angular momentum density: $m = 2$, $n = 2$

Electrically Charged Rotating Solutions?

Kleihaus, Kunz, Neemann 2005

vortex rings: $P = 0 \Longrightarrow J \neq 0$



energy density: $m = 2$, $n = 3$ angular momentum density: $m = 2$, $n = 3$

Outline

1. Monopoles in Flat Space
   - Non-Abelian Monopoles
   - Multimonopoles
   - Monopole-Antimonopole Pairs
   - Monopole-Antimonopole Systems
   - Dyons and Rotation

2. Gravitating Monopoles and Black Holes
   - Monopoles in Curved Space
   - Black Holes within Monopoles

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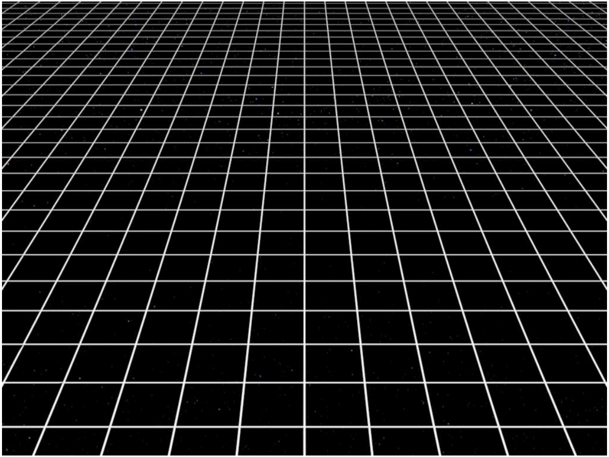
Flat Space–Time

- metric of Minkowski space-time

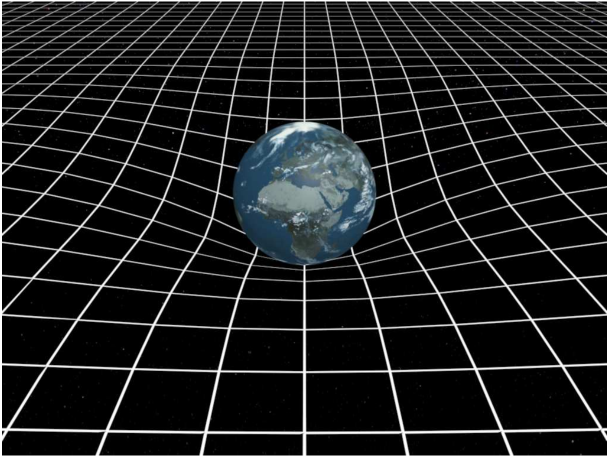
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$



Curved Space–Time

- metric of curved space-time

$$ds^2 = \; g_{\mu\nu} \; dx^\mu dx^\nu$$



Einstein Equations

- metric

$$ds^2 = \; g_{\mu\nu} \; dx^\mu dx^\nu$$

  $g_{\mu\nu}$: metric tensor

- Einstein equations

  matter tells space how to curve

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

  $G_{\mu\nu}$: Einstein tensor

  $T_{\mu\nu}$: energy-momentum tensor

Coupling Monopoles to Gravity

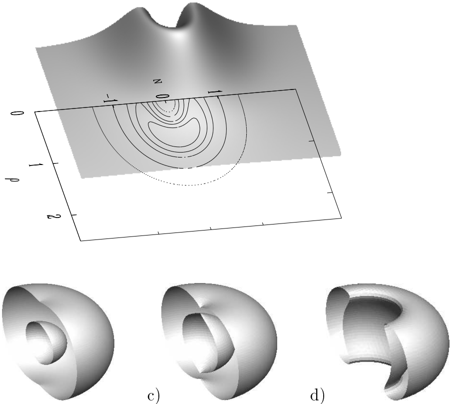
Regular solutions:

- formation of horizons
- attraction between like monopoles
- solutions with no flat space limit

Black holes:

- counterexamples to the no-hair conjecture
- non-spherical static black holes
- systems of non-abelian black holes?

$n = 2,\ \alpha = 1.0,\ \beta = 0,\ x_\Delta = 1.0$



a)

b) $\epsilon = 3.0$ c) $\epsilon = 3.5$ d) $\epsilon = 4.0$

Einstein-Yang-Mills-Higgs Theory

Einstein-Yang-Mills-Higgs action

$$\int \left\{ \underbrace{\frac{R}{16\pi G}}_{\text{gravity}} - \underbrace{\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})}_{\text{Yang-Mills (YM)}} - \underbrace{\frac{1}{4}\mathrm{Tr}\left\{D_\mu\Phi D^\mu\Phi\right\}}_{\text{Higgs field}} - \underbrace{\frac{\lambda}{8}\mathrm{Tr}\left(\Phi^2-\eta^2\right)^2}_{\text{Higgspotential}} \right\} \sqrt{-g}d^4x$$

Einstein equations

Einstein tensor $\longrightarrow \quad G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \longleftarrow$ stress–energy tensor

matter field equations

$$D_\mu F^{\mu\nu} = \frac{1}{4} ie\left[\,\Phi, D^\nu\Phi\,\right]$$

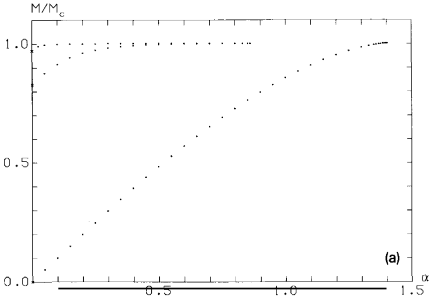
$$D_\mu D^\mu\Phi = \lambda \mathrm{Tr}\left(\Phi^2 - v^2\right)\Phi$$

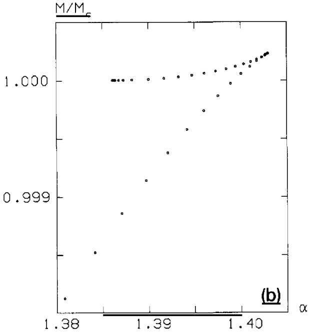
with gauge covariant derivative $D_\mu = \nabla_\mu + ie\left[A_\mu, \cdot\,\right]$

Static Spherically Symmetric EYMH Monopoles

Nieuwenhuizen, Wilkinson, Perry 1976;
Breitenlohner, Forgacs, Maison 1992; Lee, Nair, Weinberg 1992

metric: $$ds^2 = -A^2(r)N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2 d\Omega^2$$



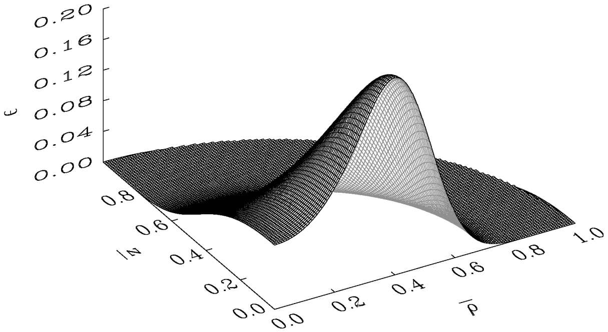


parameters $\alpha^2 = 4\pi G\eta^2,\ \lambda = e^2\beta^2$

Static Axially Symmetric EYMH Monpoles

Hartmann, Kleihaus, Kunz 2001

metric: $$ds^2 = -f(r,\theta)dt^2 + \frac{m(r,\theta)}{f(r,\theta)}\left(dr^2 + r^2 d\theta^2\right) + \frac{l(r,\theta)}{f(r,\theta)} r^2 \sin^2\theta d\varphi^2$$

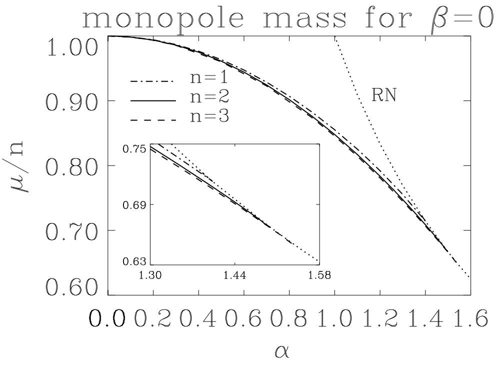


$\epsilon = -T_0^0$

Static Axially Symmetric EYMH Monpoles

Hartmann, Kleihaus, Kunz 2001

metric: $$ds^2 = -f(r,\theta)dt^2 + \frac{m(r,\theta)}{f(r,\theta)}\left(dr^2 + r^2 d\theta^2\right) + \frac{l(r,\theta)}{f(r,\theta)} r^2 \sin^2\theta d\varphi^2$$

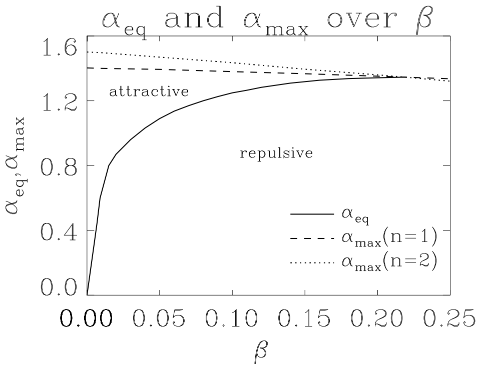


formation of horizons

Static Axially Symmetric EYMH Monpoles

Hartmann, Kleihaus, Kunz 2001

metric: $$ds^2 = -f(r,\theta)dt^2 + \frac{m(r,\theta)}{f(r,\theta)}\left(dr^2 + r^2 d\theta^2\right) + \frac{l(r,\theta)}{f(r,\theta)} r^2 \sin^2\theta d\varphi^2$$

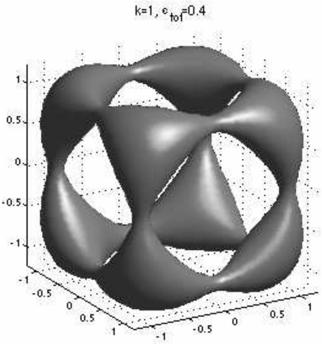


attraction of like monopoles

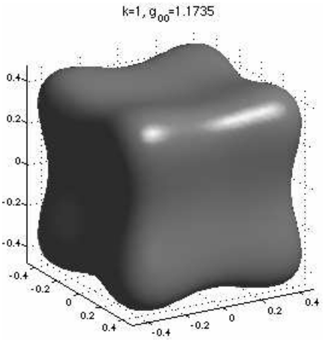
Regular Solutions with Platonic Symmetries

Kleihaus, Kunz, Myklevoll 2006

- approximate solutions



energy density

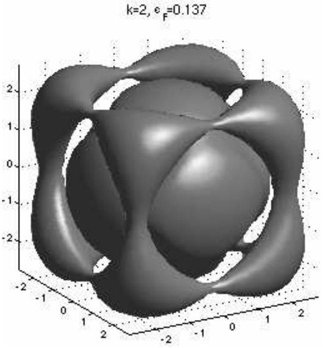


metric function g_{00}

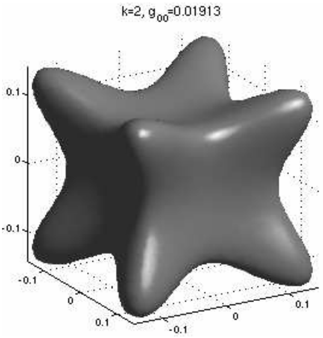
Regular Solutions with Platonic Symmetries

Kleihaus, Kunz, Myklevoll 2006

- approximate solutions



energy density



metric function g_{00}

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Einstein–Maxwell Black Holes

	static	rotating
spherically symmetric	Schwarzschild (M) Reissner-Nordström (M, Q, P)	–
axially symmetric	–	Kerr (M, J) Kerr–Newman (M, Q, P, J)

- Uniqueness theorem

  black holes are uniquely determined by their mass M, angular momentum J, charges Q and P

- Israel's theorem

  static black holes are spherically symmetric

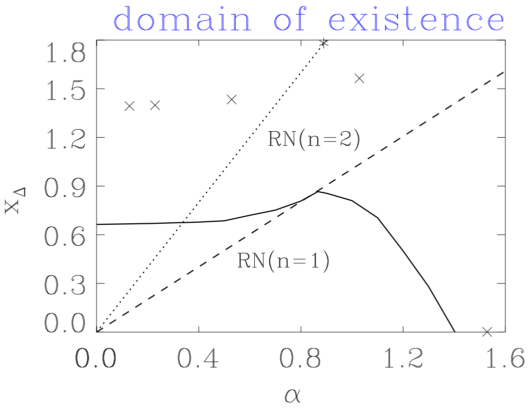
- Staticity theorem

  stationary black holes with non-rotating horizon are static

- ...

Static Black Holes within Monopoles

Breitenlohner, Forgacs, Maison 1992; Lee, Nair, Weinberg 1992
black holes within monopoles: spherical symmetry
Hartmann, Kleihaus, Kunz 2002
black holes within multimonopoles: axial symmetry



no uniqueness

Static Black Holes within Monopoles

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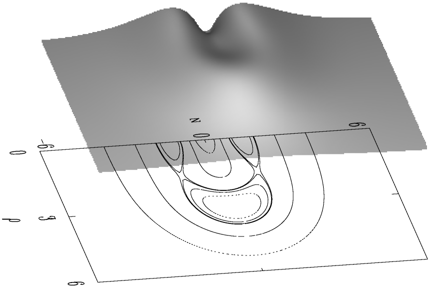

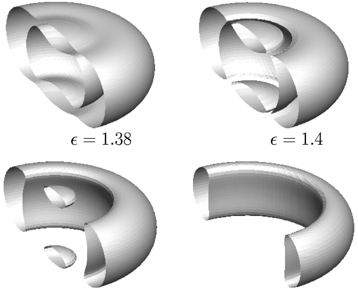

Static Black Holes within Monopoles

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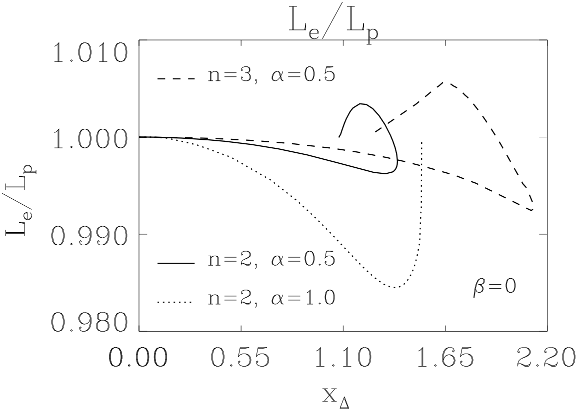
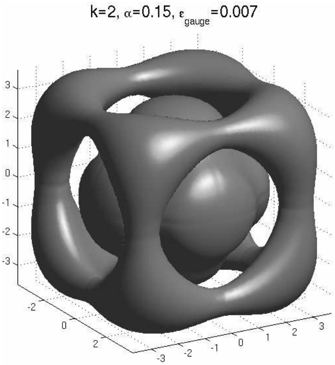


no Israel's theorem

Platonic Black Holes?

Black holes with only discrete symmetries?

- perturbative solutions
  Ridgway, Weinberg 1995
- non-perturbative solutions
  work to be done

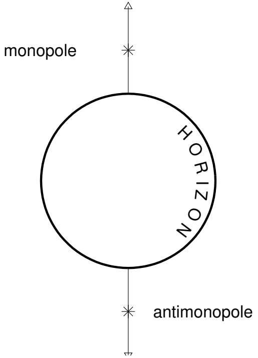


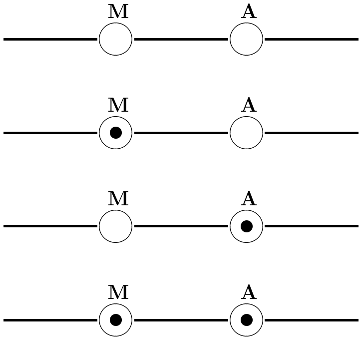
Platonic black holes?

Systems of Black Holes?

Kleihaus, Kunz 2001

MAPs with central black hole



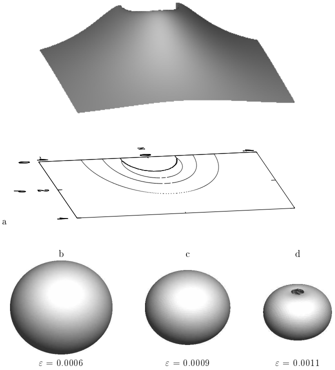


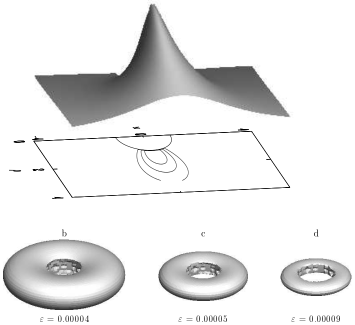
MA diholes?

work in progress

Rotating EYMH Black Holes

Kleihaus, Kunz, Navarro-Lérida 2004





slow rotation

fast rotation

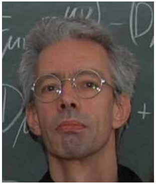
Thanks

Yves Brihaye

Betti Hartmann

Rustam Ibadov

Burkhard Kleihaus



Michael Leissner

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Ulrike Neemann

Yasha Shnir

Abha Sood

Tigran Tchrakian

Marion Wirschins