Summary and Outlook

Geodesic motion in cosmic string space-times

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Collaborations and References

Work done in collaboration with:

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References:

B. Hartmann and P. Sirimachan,
JHEP (2010) [arXiv:1007.0863 [gr-qc]].
E. Hackmann, B. Hartmann, C. Lämmerzahl and P. Sirimachan,
Phys. Rev. D **82** *(2010) 044024 [arXiv:1006.1761 [gr-qc]].*
E. Hackmann, B. Hartmann, C. Lämmerzahl and P. Sirimachan,
Phys. Rev. D **81** *(2010) 064016 [arXiv:0912.2327 [gr-qc]].*

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Outline

Summary and Outlook

Cosmic strings

- Cosmic strings form when axial symmetry gets spontaneously broken during phase transitions in the early universe
- line-like defects
  ($\rightarrow$ compare to vortices in superfluids)
- energy per unit length

$$m_{(3)} \sim T_c^2$$

  T_c: temperature of phase transition
- can be as heavy as $m_{(3)} \approx 10^{12} kg/m$

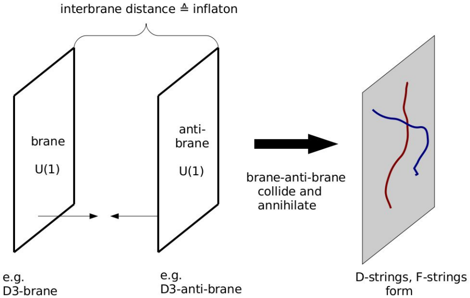
Fundamental strings (of String theory)

- Fundamental (F-) strings ...
  - have zero width
  - have tension close to the Planck scale
  - end on D-branes
  - D1-brane = D-string

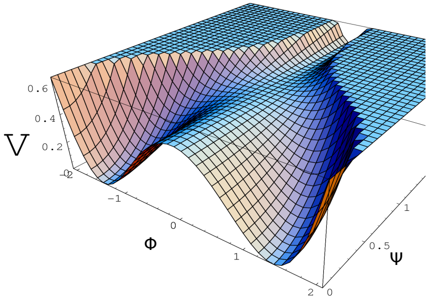
Connection between cosmic strings and fundamental strings ???

- **NO**: perturbative strings as cosmic strings ruled out (Witten, 1985)

- **YES**: cosmic strings are formed in **inflationary models** originating from string theory
  - D-, F- and bound states of p F-strings and q D-strings (p-q-strings) are formed in **brane inflation** (Jones, Stoica, Tye (2002); Sarangi, Tye (2002))



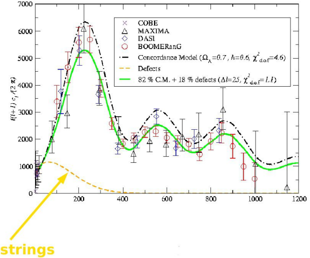
- ... and also: Hybrid inflation (Linde (1994))



- two scalar fields
- inflation ends due to **spontaneous symmetry breaking**
- **cosmic strings form generically at the end of hybrid inflation in Supersymmetric Grand Unified Theories** (Jeannerot, Rocher, Sakellariadou (2003))

Detection of cosmic strings

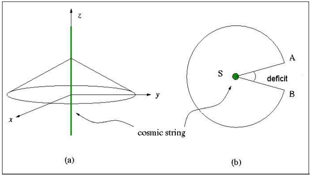
- Cosmic Microwave background data can't be explained by cosmic strings only....

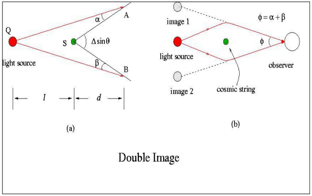


... but maybe important contribution
(e.g. Bouchet, Peter, Riazuelo, Sakellariadou (2002))

Detection of cosmic strings

- Gravitational lensing





important to understand geodesic motion of massive and massless test particles

The Geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

$\Gamma^\mu_{\rho\sigma}$ Christoffel symbol:

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right)$$

τ: *affine parameter* (proper time for time-like geodesics)

$g_{\mu\nu}$: metric tensor

Two approaches when describing cosmic string space–times

1. **macroscopic description**: Nambu-Goto action
   $\longrightarrow$ **infinitely thin strings**
   - *Advantages*: simple to treat; analytic results possible
   - *Disadvantages*: no connection to underlying field theory
2. **microscopic description**: field theoretical models
   $\longrightarrow$ **finite core width**
   - *Advantages*: “proper” description
   - *Disadvantages*: solutions only available *numerically*

Schwarzschild black hole pierced by cosmic string

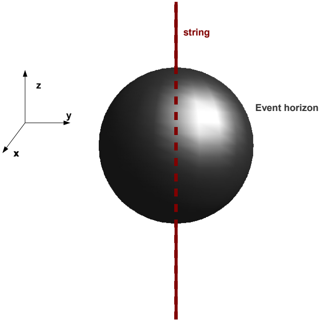
Ansatz for the metric in spherical coordinates (t, r, θ, φ)

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 \\
&+ r^2\left(d\theta^2 + \beta^2 \sin^2\theta d\varphi^2\right)
\end{aligned}
$$

$M_{\rm phys} = \beta M$ physical mass of black hole
$\delta = 2\pi(1-\beta) = 8\pi G m_{(3)} \sim 8\pi(\eta/M_{\rm Pl})^2$: deficit angle
$m_{(3)}$: energy per unit length of the string
η: symmetry breaking scale at which string forms
$M_{\rm Pl} = G^{-1/2}$: Planck mass

Schwarzschild black holes pierced by cosmic strings

Static black hole pierced by infinitely thin cosmic string



$\beta < 1$: cosmic string is long-range *hair*

$r = 2M$: event horizon

Symmetries

- **Globally** axially symmetric
- **locally** four Killing vectors

$$\xi = \frac{\partial}{\partial t}$$

$$\chi_{(1)} = \sin(\beta\varphi)\frac{\partial}{\partial\theta} + \frac{1}{\beta}\cos(\beta\varphi)\cot\theta\frac{\partial}{\partial\varphi}$$

$$\chi_{(2)} = -\cos(\beta\varphi)\frac{\partial}{\partial\theta} + \frac{1}{\beta}\sin(\beta\varphi)\cot\theta\frac{\partial}{\partial\varphi}$$

$$\chi_{(3)} = \frac{1}{\beta}\frac{\partial}{\partial\varphi}$$

Constants of motion

- Energy E

$$\xi^\mu \frac{dx^\nu}{d\tau} g_{\mu\nu} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} =: E$$

- angular momenta L_3 and L^2

$$\chi^\mu_{(i)} \frac{dx^\nu}{d\tau} g_{\mu\nu} =: L_i \quad , i = 1, 2, 3$$

with

$$L_3 = r^2 \beta \sin^2\theta \frac{d\varphi}{d\tau}$$

$$|\vec{L}|^2 \equiv L^2 = L_1^2 + L_2^2 + {L_3}^2 = r^4 \left(\frac{d\theta}{d\tau}\right)^2 + \frac{{L_3}^2}{\sin^2\theta}$$

and L_1 and L_2 are trivial

- $$\varepsilon = \frac{ds^2}{d\tau^2} = \begin{cases} -1 & \text{for massive test particles} \\ 0 & \text{for massless test particles} \end{cases}$$

Components of Geodesic equation

$$\left(\frac{dt}{d\tau}\right)^2 = E^2\left(1-\frac{2M}{r}\right)^{-2}$$

$$\left(\frac{dr}{d\tau}\right)^2 = E^2-\left(\frac{L^2}{r^2}+\varepsilon\right)\left(1-\frac{2M}{r}\right)$$

$$\left(\frac{d\theta}{d\tau}\right)^2 = \frac{L^2}{r^4}-\frac{{L_3}^2}{r^4\sin^2\theta}$$

$$\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{{L_3}^2}{\beta^2 r^4\sin^4\theta}$$

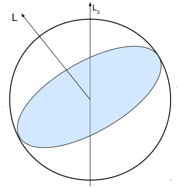
Angular motion $\theta(\varphi)$

From the θ and φ-component

$$\cot^2\theta = (k^2 - 1)\sin^2(\beta\varphi) \quad , \quad k^2 = \frac{L^2}{{L_3}^2}$$

Turning points of θ

$$\frac{d\theta}{d\tau} = 0 \Rightarrow \sin^2\theta = \frac{1}{k^2} \Rightarrow \beta\varphi = \frac{\pi}{2} + n\pi \quad , \quad n = \pm 0, \pm 1, ...$$



For $\beta \neq 1$:
Geodesic motion in precessing plane with $\vec{L}$ as normal
$\Rightarrow$ Geodesics with $\theta \neq \frac{\pi}{2}$ are not flat

Radial motion $r(\theta)$ and $r(\varphi)$

New coordinate $z = \frac{2M}{r} - \frac{1}{3}$, $z \in [-\frac{1}{3} : \infty)$

$$\frac{dz}{\sqrt{P(z)}} = \frac{1}{2}\left(1 - \frac{1}{k^2 \sin^2\theta}\right)^{-1/2} d\theta$$

$$\frac{dz}{\sqrt{P(z)}} = \frac{1}{2}\beta k \frac{1}{(k^2-1)\sin^2(\beta\varphi) + 1} d\varphi$$

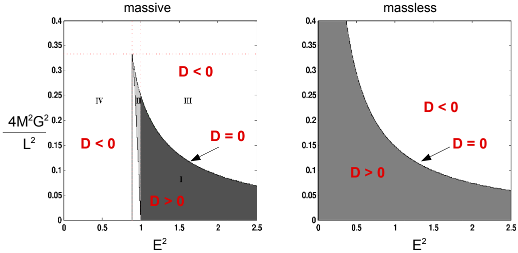
with

$$P(z) = 4z^3 - 4\left[\frac{1}{3} - \left(\frac{4M^2}{L^2}\right)\varepsilon\right] z - 4\left[\frac{2}{27} + \frac{2}{3}\left(\frac{4M^2}{L^2}\right)\varepsilon - \frac{4G^2M^2}{L^2}E^2\right]$$
$$= 4z^3 - g_2 z - g_3$$

Classification of solutions

- Need $P(z) > 0$ to have solutions $\Rightarrow$ study roots of $P(z)$
- Discriminant D with

$$D = g_2^3 - 27 g_3^2 \begin{cases} > 0 & \text{three real roots } e_1 > e_2 > e_3 \\ < 0 & \text{one real root} \\ = 0 & \text{either one or two real roots} \end{cases}$$



Classification of solutions

- Effective potential

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff}(r) = \frac{E^2 - \varepsilon}{2}$$

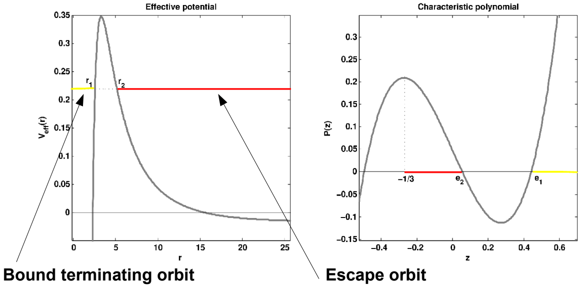
with

$$V_{\rm eff}(r) = -\varepsilon\frac{M}{r} + \frac{L^2}{2r^2} - \frac{L^2 M}{r^3}$$

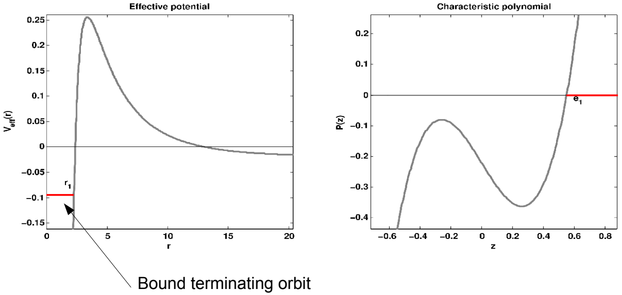
- Turning points of r correspond to $P(z) = 0$

$$\frac{L^2}{32M^2}P(z(r)) = \frac{E^2 - \varepsilon}{2} - V_{\rm eff}(r)$$

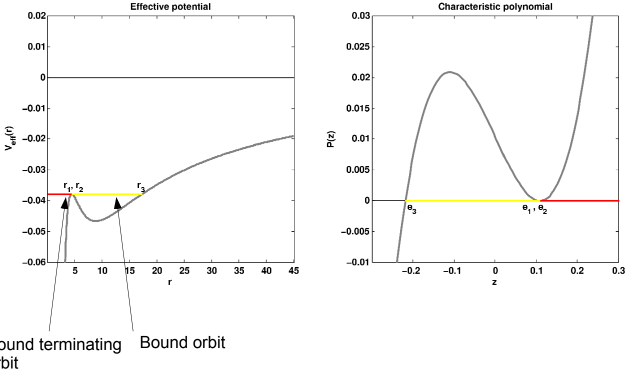
Example for $D > 0$



Example for $D < 0$



Example for $D = 0$



Solutions to the geodesic equation

In terms of Weierstrass $\wp$-function

$$r(\theta) = \frac{2M}{\wp(g(\theta) - \mathrm{c}) + \frac{1}{3}}$$

$$r(\varphi) = \frac{2M}{\wp(f(\varphi) - \mathrm{c}) + \frac{1}{3}}$$

with

$$g(\theta) \equiv \frac{1}{2}\left[\arcsin\left(\frac{\cos\theta}{\sqrt{1-k^{-2}}}\right) - \arcsin\left(\frac{\cos\theta_0}{\sqrt{1-k^{-2}}}\right)\right]$$

$$f(\varphi) \equiv -\frac{1}{2}\arctan\left[k\tan(\beta(\varphi-\varphi_0))\right]$$

Solutions to the geodesic equation

Value of constant c:

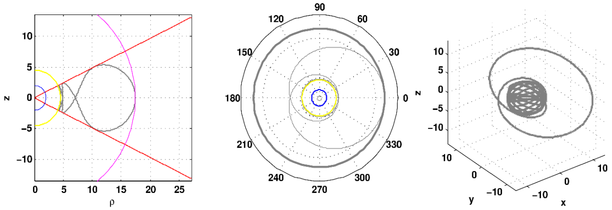
$$c = \int_{z_0}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

- $z_0 = \infty \Rightarrow c = 0$
- $z_0 = e_1 \Rightarrow c = \frac{K(\mathcal{K})}{\sqrt{e_1 - e_3}}$
- $z_0 = e_2 \Rightarrow c = \frac{K(\mathcal{K})}{\sqrt{e_1 - e_3}} + i\frac{K(\mathcal{K}')}{\sqrt{e_1 - e_3}}$
- $z_0 = e_3 \Rightarrow c = i\frac{K(\mathcal{K}')}{\sqrt{e_1 - e_3}}$

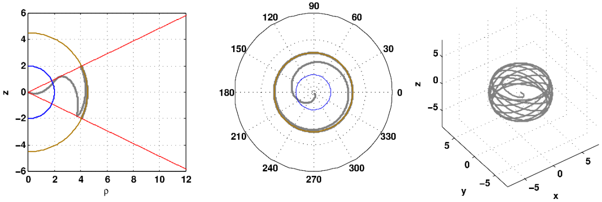
with K: complete elliptic integral of 1.kind with modulus

$$\mathcal{K} = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}} \; , \quad \mathcal{K}' = \sqrt{1 - \mathcal{K}^2}$$

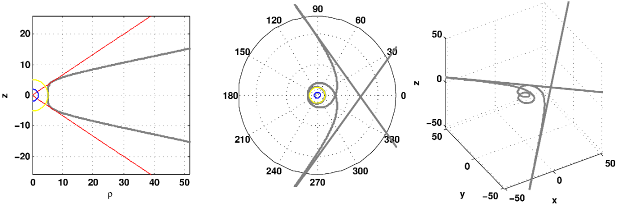
Example of geodesic: bound orbit



Example of geodesic: bound terminating orbit



Example of geodesic: escape orbit



Light deflection

For $k = 1$ and **massless** test particles deflection angle

$$\Delta\varphi = \frac{1}{\beta}\left[\frac{4}{\sqrt{e_1 - e_3}}\int_0^{\varphi_c}\frac{d\varphi}{\sqrt{1-\mathcal{K}^2\sin^2(\varphi)}} + 2\omega_1\right] + \pi\left(\frac{1}{\beta} - 1\right)$$

ω_1: first half period

Observational constraints with $(\Delta\varphi)_S$ Schwarzschild value

$$\frac{\Delta\varphi - (\Delta\varphi)_S}{(\Delta\varphi)_S} \lesssim 10^{-5}$$

$$\Rightarrow \quad (1-\beta) \lesssim 10^{-11} \quad \Rightarrow \quad m_{(3)} \lesssim 10^{16}\,\frac{\mathrm{kg}}{\mathrm{m}}$$

Perihelion shift

For $k = 1$ and **massive** test particles perihelion shift

$$\Delta\varphi = \frac{4}{\beta} \frac{K(\mathcal{K})}{\sqrt{e_1 - e_3}} - 2\pi$$

Observational constraints with $(\Delta\varphi)_S$ Schwarzschild value

$$\frac{\Delta\varphi - (\Delta\varphi)_S}{(\Delta\varphi)_S} \lesssim 10^{-4}$$

$$\Rightarrow \quad (1-\beta) \lesssim 10^{-10} \quad \Rightarrow \quad m_{(3)} \lesssim 10^{17} \frac{\mathrm{kg}}{\mathrm{m}}$$

Kerr black hole pierced by cosmic string

Ansatz for the metric in Boyer-Lindquist coordinates (t, r, θ, φ)

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\
&+ \beta^2\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2 - \beta\frac{4Mra\sin^2\theta}{\rho^2}dtd\varphi
\end{aligned}
$$

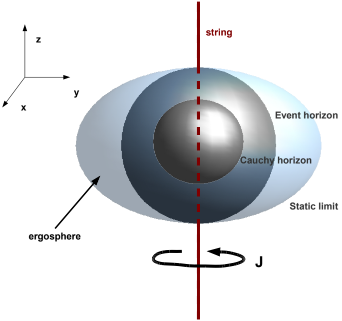
with

$$\rho^2 = r^2 + a^2\cos^2\theta \ , \quad \Delta = r^2 - 2Mr + a^2$$

$a = J/M$: angular momentum J per mass M
$\delta = 2\pi(1-\beta) = 8\pi G m_{(3)} \sim 8\pi(\eta/M_{\rm Pl})^2$: deficit angle
$m_{(3)}$: energy per unit length of the string
η: symmetry breaking scale at which string forms
$M_{\rm Pl}$: Planck mass

Kerr black hole pierced by cosmic string

Rotating black hole pierced by infinitely thin cosmic string



$\beta < 0$: cosmic string is long-range *hair*

$r_+ = M + \sqrt{M^2 - a^2}$: event horizon, $r_- = M - \sqrt{M^2 - a^2}$: Cauchy horizon

$2Mr = \rho^2$: static limit

Boyer-Lindquist coordinates

Relation to cartesian coordinates

$$
\begin{aligned}
x &= \sqrt{r^2 + a^2} \sin\theta \cos(\beta\varphi) \\
y &= \sqrt{r^2 + a^2} \sin\theta \sin(\beta\varphi) \\
z &= r \cos\theta
\end{aligned}
$$

- $r = 0 \Rightarrow$ disc with deficit angle $\delta = 2\pi(1 - \beta)$
- *Physical singularity* at $r = 0$, $\theta = \pi/2$
  $\Rightarrow$ ring with deficit angle $\delta = 2\pi(1 - \beta)$
- $r < 0$: another conical space-time without horizons

Constants of motion

- Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$ $\Rightarrow$ conserved quantities

$$\left(1-\frac{2Mr}{\rho^2}\right)\frac{dt}{d\tau}+\beta\frac{2Mar}{\rho^2}\sin^2\theta\frac{d\varphi}{\tau}=: E$$

$$-\frac{2Mar}{\rho^2}\sin^2\theta\frac{dt}{d\tau}+\beta\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta\frac{d\varphi}{d\tau}=: L_z$$

- Carter constant K: separability of Hamilton-Jacobi equations
-

$$\varepsilon=\frac{ds^2}{d\tau^2}=\begin{cases}-1 & \text{massive particles}\\ 0 & \text{massless particles}\end{cases}$$

Hamilton-Jacobi equations

Hamilton–Jacobi equations

$$\frac{\partial S}{\partial \tau} = \frac{1}{2} g^{\mu\nu} \left(\partial_\mu S\right) \left(\partial_\nu S\right)$$

S: Hamilton function with Ansatz

$$S = \frac{1}{2}\varepsilon\tau - Et + \beta L_z \varphi + S_r(r) + S_\theta(\theta)$$

$S_r(r)$: function of r only

$S_\theta(\theta)$: function of θ only

Components of geodesic equation

Introduce **Mino time** $d\lambda = \frac{d\tau}{\rho^2}$

$$\frac{dr}{d\lambda} = \pm\sqrt{R(r)}$$

$$\frac{d\theta}{d\lambda} = \pm\sqrt{\Theta(\theta)}$$

$$\frac{d\varphi}{d\lambda} = \frac{1}{\beta}\left(\frac{L_z \csc^2\theta - aE}{\sqrt{\Theta(\theta)}}\frac{d\theta}{d\lambda} + \frac{aP(r)}{\Delta(r)\sqrt{R(r)}}\frac{dr}{d\lambda}\right)$$

$$\frac{dt}{d\lambda} = \frac{a(L_z - aE\sin^2\theta)}{\sqrt{\Theta(\theta)}}\frac{d\theta}{d\lambda} + \frac{(r^2+a^2)P(r)}{\Delta(r)\sqrt{R(r)}}\frac{dr}{d\lambda}$$

with

$$\Theta(\theta) = K - (L_z - aE)^2 - \cos^2\theta\left(L_z{}^2\csc^2\theta - a^2(E^2+\varepsilon)\right)$$

$$P(r) = E(r^2+a^2) - L_z a$$

$$R(r) = P(r)^2 - \Delta(r)\left(K - \varepsilon r^2\right)$$

$r(\lambda)$ motion

- Need $R(r) > 0$ to have solutions with $R(r) = 0$ *turning points* of $r(\lambda)$ motion
- $R(r) = 0 \Rightarrow$ (a) 4 real, (b) 2 real & 2 complex, (c) 4 complex roots
- new coordinate

$$z = \frac{1}{c_2}\left(\frac{1}{r - r_1} - c_1\right)$$

where r_1 largest real root and c_1 and c_2 depend on K, L_z, E, ε

$$\Rightarrow d\lambda = \frac{dr}{\sqrt{R(r)}} = -\frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

where g_2 and g_3 depend on K, L_z, E, ε

$r(\lambda)$ motion

Solution in terms of Weierstrass $\wp$-function

$$r(\lambda) = \left(\frac{1}{c_2 \wp\left(\frac{1}{c_2}(\lambda - \lambda_0) + C; g_2, g_3\right) + c_1} + r_1 \right)$$

with

$$C = \int_{z_0}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$$

- $z_0 = \infty \Rightarrow C = 0$
- $z_0 = e_1 \Rightarrow C = K(\mathcal{K})/\sqrt{e_1 - e_3}$
- $z_0 = e_2 \Rightarrow C = K(\mathcal{K})/\sqrt{e_1 - e_3} + iK(\mathcal{K}')/\sqrt{e_1 - e_3}$

$\theta(\lambda)$ motion

- Need $\Theta(\theta) > 0$ to have solutions with $\Theta(\theta) = 0$ *turning points* of $\theta(\lambda)$ motion
- new coordinate

$$\tilde{z} = \frac{1}{c_3}\left(\sec^2\theta - c_4\right)$$

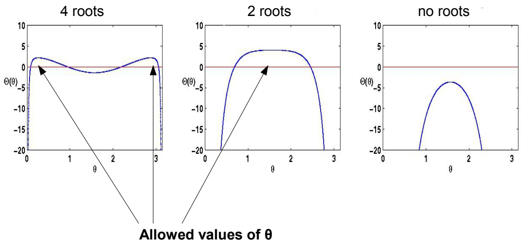
where c_3 and c_4 depend on K, L_z, E, ε

$$\Rightarrow d\lambda = \frac{d\theta}{\sqrt{\Theta(\theta)}} = \frac{d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}}$$

where $\tilde{g}_2$ and $\tilde{g}_3$ depend on K, L_z, E, ε

$\theta(\lambda)$ motion

- Discriminant $D = \tilde{g}_2^3 - 27\tilde{g}_3^2 > 0 \Rightarrow$ 3 real roots $\tilde{e}_i$, $i = 1, 2, 3$
- BUT: roots might not fulfill $(c_3\tilde{z} + c_4)^{-1} = \cos^2\theta \leq 1$
- AND: for one root $\Rightarrow$ two values of θ with $\cos\theta = \pm(c_3\tilde{z} + c_4)^{-1/2}$
- $\Theta(\theta) = 0 \Rightarrow$ (a) 4 real, (b) 2 real, (c) no real roots



$\theta(\lambda)$ motion

Solution in terms of Weierstrass $\wp$-function

$$\theta(\lambda) = \arccos\left[\pm\frac{1}{\sqrt{c_3\wp\left(\frac{1}{c_3}(\lambda-\lambda_0)+\tilde{C};\tilde{g}_2,\tilde{g}_3\right)+c_4}}\right]$$

with

$$\tilde{C} = \int_{\tilde{z}_0}^{\infty}\frac{d\tilde{z}}{\sqrt{4\tilde{z}^3-\tilde{g}_2\tilde{z}-\tilde{g}_3}}$$

- $\tilde{z}_0 = \tilde{e}_1 \Rightarrow \tilde{C} = K(\mathcal{K})/\sqrt{\tilde{e}_1-\tilde{e}_3}$
- $\tilde{z}_0 = \tilde{e}_3 \Rightarrow \tilde{C} = iK(\mathcal{K}')/\sqrt{\tilde{e}_1-\tilde{e}_3}$

$\varphi(\lambda)$ and $t(\lambda)$ motion

Rewrite

$$\beta d\varphi = \frac{L_z \csc^2\theta - aE}{\sqrt{\Theta(\theta)}} d\theta + \frac{a\Delta^{-1}P(r)}{\sqrt{R(r)}} dr =: dI_\theta + dI_r$$

$$dt = a(L_z - aE\sin^2\theta)\frac{d\theta}{\sqrt{\Theta(\theta)}} + (r^2+a^2)\Delta(r)^{-1}P(r)\frac{dr}{\sqrt{R(r)}} =: d\bar{I}_\theta + d\bar{I}_r$$

solutions for I_θ, I_r, $\bar{I}_\theta$, $\bar{I}_r$ in terms of Weierstrass ζ- and σ-functions

Example: Solution for I_θ

With new coordinate $\tilde{z} = \left(\sec^2\theta - c_4\right)/c_3$:

$$dI_\theta = \frac{c_3(L_z - aE)d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}} + \frac{L_z(c_4 + c_3c_5)d\tilde{z}}{(\tilde{z} - c_5)\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}} \ , \quad c_5 = (1 - c_4)/c_3$$

Introducing

$$x := \int_\infty^{\tilde{z}} \frac{d\tilde{z}}{\sqrt{4\tilde{z}^3 - \tilde{g}_2\tilde{z} - \tilde{g}_3}} \quad \Rightarrow \tilde{z} = \wp(x; \tilde{g}_2, \tilde{g}_3)$$

we have

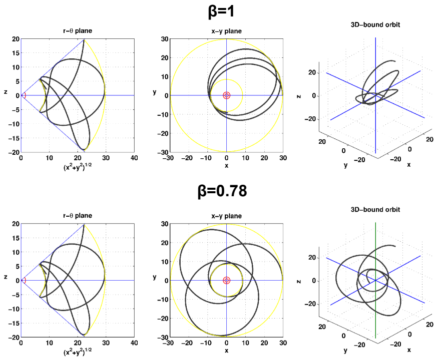
$$dI_\theta = c_3(L_z - aE)dx + (c_4 + c_3c_5)\frac{dx}{\wp(x) - c_5}$$

with $x = \lambda/c_3$ we find

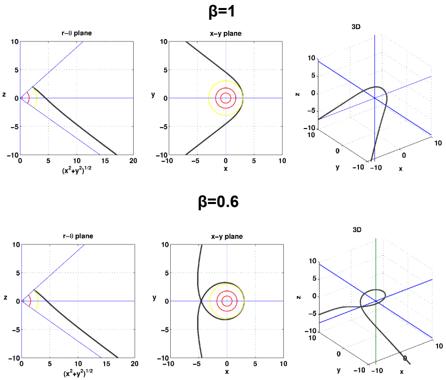
$$I_\theta = (L_z - aE)\lambda + \sum_{i=1}^{2} \frac{c_4 + c_3c_5}{\wp'(x_i)}\left[\frac{\lambda}{c_3}\zeta(x_i) + \ln\left(\sigma(x - x_i)\right)\right]$$

where $\wp(x_i) = c_5$, $i = 1, 2$

Example of geodesic: bound orbit



Example of geodesic: escape orbit



Lense-Thirring effect

- Frame dragging effect of rotating massive body
- LAGEOS satellites:
  $\Omega_{\rm LT}(\beta = 1) \approx 39 \cdot 10^{-3}$ arcseconds/year (10% accuracy)
- If cosmic string present, i.e. $\beta \neq 1$:

$$\Omega_{\rm LT}(\beta \neq 1) - \Omega_{\rm LT}(\beta = 1) \leq 4 \cdot 10^{-3} \mathrm{arcseconds/year}$$

- bound on energy per unit length $m_{(3)}$ of cosmic string

$$\frac{1}{\beta} - 1 \lesssim 10^{-11} \Rightarrow m_{(3)} \lesssim 10^{16}\ \mathrm{kg/m}$$

Abelian-Higgs strings

$U(1)$ Abelian-Higgs model minimally coupled to gravity:

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + \mathcal{L} \right)$$

with matter Lagrangian

$$\mathcal{L} = D_\mu \phi (D^\mu \phi)^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} \left(\phi\phi^* - \eta^2 \right)^2$$

with

$$D_\mu \phi = \nabla_\mu \phi - ieA_\mu \phi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

ϕ: complex scalar field
A_μ : U(1) gauge field
e: gauge coupling
λ: self-interaction coupling

$\eta \neq 0$: vacuum expectation value

Ansatz for static, straight strings

- Matter fields (Nielsen & Olesen, 1973)

$$\phi(\rho,\varphi) = \eta h(\rho) e^{in\varphi} \quad , \quad A_\mu dx^\mu = \frac{1}{e}\left(n - P(\rho)\right) d\varphi$$

  n: degree of map $S^1 \to S^1$, homotopy group $\pi_1(S^1) = \mathbb{Z}$

- Metric

$$ds^2 = N^2(\rho)dt^2 - d\rho^2 - L^2(\rho)d\varphi^2 - N^2(\rho)dz^2$$

Four non-linear coupled 2nd order ordinary differential equations in h, P, N and L $\Rightarrow$ have to be solved *numerically*

Equations

$$
\begin{aligned}
\frac{(N^2 L h')'}{N^2 L} &= \frac{P^2 h}{L^2} + \frac{\alpha}{2} h(h^2 - 1) \\
\frac{L}{N^2}\left(\frac{N^2 P'}{L}\right)' &= 2h^2 P \\
\frac{(LNN')'}{N^2 L} &= \gamma\left[\frac{(P')^2}{2L^2} - \frac{\alpha}{4}\left(h^2 - 1\right)^2\right] \\
\frac{(N^2 L')'}{N^2 L} &= -\gamma\left[\frac{2h^2 P^2}{L^2} + \frac{(P')^2}{2L^2} + \frac{\alpha}{4}\left(h^2 - 1\right)^2\right] .
\end{aligned}
$$

with

$$\gamma = 8\pi G\eta^2 = 8\pi \frac{\eta^2}{M_{\rm Pl}^2} \;\; , \;\; \alpha = \frac{\lambda}{e^2} = \frac{M_{\rm H}^2}{M_{\rm W}^2}$$

$M_{\rm H} = \sqrt{2\lambda}\eta$ Higgs boson mass

$M_{\rm W} = \sqrt{2}e\eta$ gauge boson mass

Boundary conditions

- Regularity at the origin

$$
\begin{aligned}
h(0) &= 0 \ , \ P(0) = n \ , \ N(0) = 1, \\
N'(0) &= 0, \ L(0) = 0 \ , \ L'(0) = 1
\end{aligned}
$$

- Finiteness of energy

$$
h(\infty) = 1 \ , \ P(\infty) = 0
$$

Properties of Abelian–Higgs strings

- magnetic field $\vec{B} = B_z \vec{e}_z$ and quantized magnetic flux:

$$B_z = -\frac{1}{e}\frac{dP/d\rho}{\rho} \quad , \quad \Phi_M = -\frac{2\pi n}{e}$$

- scalar core width $\sim (\text{Higgs mass})^{-1} = M_H^{-1} = (\sqrt{2\lambda}\eta)^{-1}$
- width of flux tubes
  $\sim (\text{gauge boson mass})^{-1} = M_W^{-1} = (\sqrt{2}e\eta)^{-1}$
- $M_H = M_W$: saturate energy bound $m_{(3)} = 2\pi\eta^2 n$
  $\Rightarrow$ **BPS limit**, but **no** analytic solutions

Geodesics: Constants of motion

$$E := N^2 \frac{dt}{d\tau} \quad \text{energy}$$

$$L_z := L^2 \frac{d\varphi}{d\tau} \quad \text{angular momentum}$$

$$p_z := N^2 \frac{dz}{d\tau} \quad \text{momentum}$$

$$\varepsilon = \frac{ds^2}{d\tau^2} = \begin{cases} 1 & \text{for massive test particles} \\ 0 & \text{for massless test particles} \end{cases}$$

Geodesic equation

$$\frac{1}{2}\left(\frac{d\rho}{d\tau}\right)^2 = \bar{E} - V_{\rm eff}(\rho)$$

with

$$\bar{E} = (E^2 - \varepsilon)$$

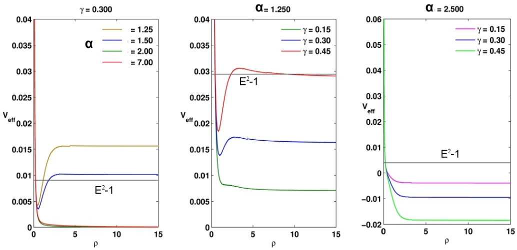
and

$$V_{\rm eff}(\rho) = \frac{1}{2}\left[E^2\left(1 - \frac{1}{N^2}\right) + \frac{{p_z}^2}{N^2} + \frac{{L_z}^2}{L^2}\right]$$

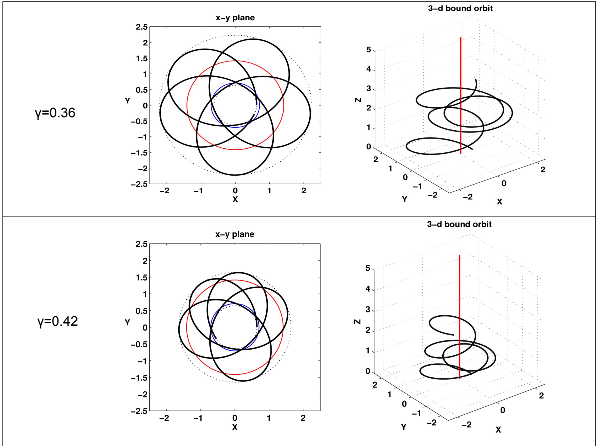
$V_{\rm eff}$: effective potential

Massive particles: Effective potential

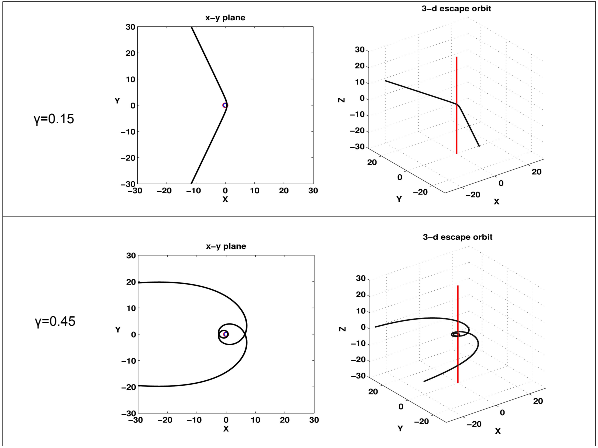
- infinite potential barrier for $L_z \neq 0$
- **no** bound orbits for $\alpha \geq 2$



Massive particles: Example of bound orbit



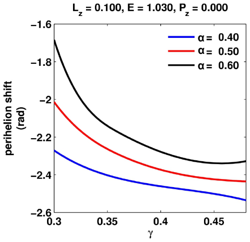
Massive particles: Example of escape orbit



Perihelion shift

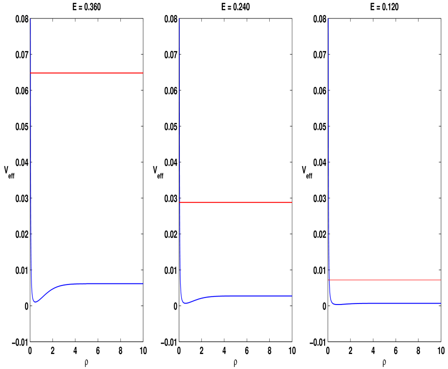
For planar motion ($p_z = 0$):

$$\Delta\varphi = 2\int_{\rho_{\min}}^{\rho_{\max}} \frac{L_z d\rho}{L(\rho)^2\left(\frac{E^2}{N(\rho)^2} - \frac{L_z{}^2}{L(\rho)^2} - 1\right)^{1/2}} - 2\pi$$



Massless particles: Effective potential

- infinite potential barrier for $L_z \neq 0$
- **no bound orbits**



Massless particles: no bound orbits

Compare to Gibbons, 1993:

In a general cosmic string space-time with topology $\mathbb{R}^2 \times \Sigma$ where Σ has positive Gaussian curvature a massless test particle must move on a geodesic that escapes to infinity in both directions.

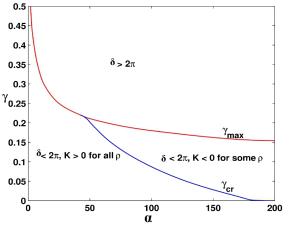
Massless particles: no bound orbits

Massless particles in x-y-plane

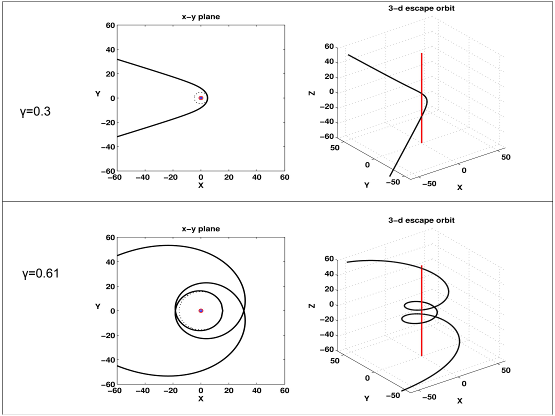
$$dt^2 = \frac{1}{N^2} d\rho^2 + \frac{L^2}{N^2} d\varphi^2 = \tilde{g}_{ij} dx^i dx^j \quad , \quad i = 1, 2$$

$\tilde{g}_{ij}$ optical metric of manifold Σ with Gaussian curvature

$$K = \frac{L'}{L} N' N - \frac{L''}{L} N^2 - (N')^2 + NN''$$



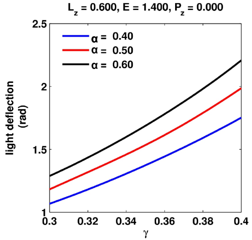
Massless particles: Example of escape orbit



Light deflection

For planar motion ($p_z = 0$):

$$\Delta\varphi = 2 \int_{\rho_{\rm min}}^{\infty} \frac{L_z d\rho}{L(\rho)^2 \left(\frac{E^2}{N(\rho)^2} - \frac{L_z{}^2}{L(\rho)^2}\right)^{1/2}} - \pi$$



Summary

- Link between cosmic strings $\leftrightarrow$ fundamental strings
- possible observation ...
  - ... in the Cosmic Microwave background (Power- and Polarization spectrum)
  - **... through motion of test particles in cosmic string space-times**
- in view of this ...
  - ... found the complete set of solutions to the geodesic equation in space-time of Schwarzschild- and Kerr black hole pierced by infinitely thin cosmic string
  - ... found solutions to the geodesic equation in the space-time of an Abelian-Higgs string

Summary

- Applications
  - computation of gravitational wave templates for extreme mass ratio inspirals
  - gravitational lensing
  - test particle motion in solar system if sun is not perfectly spherically symmetric
  - possible explanation of the observed alignment of polarization vectors of quasars on cosmological scales via remnants of cosmic string decay

Outlook

Work in progress...

- ... solutions to the geodesic equation in other numerically given space-times (semilocal, p-q-strings, superconducting...)
- ... solutions to the geodesic equation in space-time with cosmic string and (positive or negative) cosmological constant $\Rightarrow$ hyperelliptic integrals

  $\Rightarrow$ *compare Talks by C. Lämmerzahl and V. Kagramanova*