Continuous and Discrete Neumann systems on Stiefel varieties and matrix Jacobi–Mumford systems

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Classical Neumann system on $T^*S^{n-1} = \{\langle q, q\rangle = 1,\ \langle q, p\rangle = 0\} \subset \mathbb{R}^{2n}$

$$q = (q_1 \dots, q_n)^T, \quad p = (p_1 \dots, p_n)^T$$

$$H = \frac{1}{2}\langle p, p\rangle + \frac{1}{2}\langle Aq, q\rangle, \qquad A = \mathrm{diag}(a_1, \dots, a_n),$$

Hamilton equations

$$\dot{q} = p, \quad \dot{p} = -Aq + \nu q, \qquad \nu = \langle p, p\rangle - \langle q, Aq\rangle$$

First integrals in involution (Ulenbeck)

$$F_i = q_i^2 + \sum_{j \neq i} \frac{(p_i q_j - p_j q_i)^2}{a_j - a_i}, \quad i = 1, \dots, n$$

• Integrability by the Liouville theorem: *generic invariant manifolds are* $\mathbb{T}^{n-1}$ *with straight line flows on them.*

The Neumann system on T^*S^{n-1}: the Lax reprentations

Big $(n \times n)$ Lax pair was found by Moser (1983),
Small (2×2) Lax pair by Mumford (1984):

$$\dot{L}(\lambda) = [L(\lambda), \mathcal{A}(\lambda)],$$

$$L(\lambda) = \begin{pmatrix} \sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} & \sum_{i=1}^n \frac{q_i^2}{\lambda - a_i} \\ 1 - \sum_{i=1}^n \frac{p_i^2}{\lambda - a_i} & -\sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \sum_{i=1}^n \frac{\mathcal{N}_i}{\lambda - a_i},$$

$$\mathcal{A}(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + \nu(p, q) & 0 \end{pmatrix}$$

- Spectral curve $\{|a(\lambda)L(\lambda) - \mu \mathbf{I}| = 0\}$ is hyperelliptic of genus $g = n - 1$

$$\Gamma = \{\mu^2 = \underbrace{(\lambda - a_1) \cdots (\lambda - a_n)}_{a(\lambda)} (\lambda - c_1) \cdots (\lambda - c_{n-1})\}, \quad c_i = \text{const}$$

- Real generic tori $\mathbb{T}^{n-1}$ are extended to complex tori $\mathbb{T}_{\mathbb{C}}^{n-1} = \text{Jac}(\Gamma)$, the Jacobian variety of Γ

Relation with the (odd) Jacobi–Mumford systems

$$a(\lambda)L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix},$$

$$\Gamma = \{\mu^2 = R(\lambda)\},$$

$$R(\lambda) = U(\lambda)W(\lambda) + V^2(\lambda) \equiv \lambda^{2g+1} + r_1\lambda^{2g} + \cdots + r_{2g+1},$$

$$U(\lambda) = \lambda^g + u_1\lambda^{g-1} + \cdots + u_g = (\lambda - \lambda_1)\cdots(\lambda - \lambda_g),$$

$$V(\lambda) = v_1\lambda^{g-1} + \cdots + v_g,$$

$$W(\lambda) = \lambda^{g+1} + w_0\lambda^g + w_1\lambda^{g-1} + \cdots + w_g$$

- The Lax pair defines a flow (*Jacobi–Mumford system*) on

$$\mathcal{E}_g = \mathbb{C}^{3g+1}[u_1, \ldots, v_1, \ldots, w_g] \xrightarrow{\text{Jac}(\Gamma)/\Theta} \mathbb{C}^{2g+1}[r_1, \ldots, r_{2g+1}]$$

The points $P_1 = (\lambda_1, \mu_1), \ldots, P_g = (\lambda_g, \mu_g) \in \Gamma$ such that $U(\lambda_i) = 0,\ V(\lambda_i) = \mu_i$ define an point on $\text{Jac}(\Gamma)/\Theta$.

- The flow on each $\text{Jac}(\Gamma)$ is translationally invariant.

Discrete Neumann system (Bäcklund transformation)

Implicit maps $\mathcal{B}_{\lambda^*} : (p, q) \mapsto (\tilde{p}, \tilde{q})$, $\lambda^* \in \mathbb{C}$ being arbitrary step parameter.
(A. Veselov, V. Kuznetsov, P. Vanhaecke, Yu. Suris)

$$\boxed{\tilde{q} = A^{-1/2}(\lambda^*)(\beta q + p), \quad \tilde{p} = -A^{1/2}(\lambda^*)q + A^{-1/2}(\lambda^*)(\beta^2 q + \beta p),}$$

$$A(\lambda^*) = \lambda^* \mathbf{I} - A, \qquad \beta = \langle \tilde{q}, A^{1/2}(\lambda^*)q \rangle, \quad A = \mathrm{diag}(a_1, \ldots, a_n)$$

$(1/\lambda^*) \to 0$ gives the continuous limit.

To evaluate $\mathcal{B}_{\lambda^*}$, we solve the quadratic equation w.r.t. β

$$\langle q, A^{-1}(\lambda^*)q \rangle \, \beta^2 + 2\langle p, A^{-1}(\lambda^*)q \rangle \, \beta + \langle p, A^{-1}(\lambda^*)p \rangle - 1 = 0$$

Theorem

1). Up to the action of the group of reflections
$(p_i, q_i) \to (-p_i, -q_i)$, $i = 1, \ldots, n$,
the map $\mathcal{B}_{\lambda^*}$ *is equivalent to the discrete Lax pair*

$$\tilde{L}(\lambda)\, M(\lambda|\lambda^*) = M(\lambda|\lambda^*)\, L(\lambda),$$

$$M(\lambda|\lambda^*) = \begin{pmatrix} -\beta & 1 \\ \lambda - \lambda^* + \beta^2 & -\beta \end{pmatrix}, \quad \beta = \langle \tilde{q}, A^{1/2}(\lambda^*) q \rangle$$

hence, it has the same integrals as the continuous system.

2) (A. Veselov) $\mathcal{B}_{\lambda^*}$ *is given by a shift on* $Jac(\Gamma)$ *by*

$$T = \mathcal{A}(\mathcal{P}) \equiv \int_{\infty}^{\mathcal{P}} (\omega_1, \ldots, \omega_g)^T, \quad \mathcal{P} = (\lambda^*, \pm\mu^*) \in \Gamma.$$

The Stiefel variety $V(n, r) = SO(n)/SO(n-r)$ $(r < n)$

The set of $n \times r$ matrices

$$X = (e_1 \cdots e_r), \quad e_s \in \mathbb{R}^n, \quad X^T X = \mathbf{I}_r\,.$$

The cotangent bundle $T^* V(n, r)$, the set of $n \times r$ pairs (X, P), $P = (p_1 \cdots p_r)$, $p_s \in \mathbb{R}^n$ such that

$$X^T X = \mathbf{I}_r, \quad X^T P + P^T X = 0$$

The Hamilton equations (with $r \times r$ symmetric matrix multipliers Π, Λ)

$$\dot{X} = \frac{\partial H}{\partial P} - X\Pi,$$
$$\dot{P} = -\frac{\partial H}{\partial X} + X\Lambda + P\Pi.$$

The Neumann systems on $T^*V(n,r)$

Family of $SO(n)$-invariant metrics on $V(n,r)$ given by

$$T_\kappa(X,P) = \frac{1}{2}\left(\mathrm{Tr}(P^TP) - (1-\kappa)\,\mathrm{Tr}((X^TP)^2)\right).$$

Choose $H = T_\kappa + \frac{1}{2}\,\mathrm{Tr}(X^TAX)$.

- The Neumann system with the *normal* metric ($\kappa = 0$)

$$\dot X = P - XP^TX, \quad \dot P = AX + X\Lambda + P\,X^TP,$$

- The Neumann system with the *Euclidean* metric ($\kappa = 1$)

$$\dot X = P, \quad \dot P = AX + X\Lambda,$$
$$\Lambda = -X^TAX - P^TP \in \mathrm{Symm}(r \times r)$$

Both preserve the $so(r)$-momentum integral $\Psi = X^TP - P^TX$ and satisfy the $n \times n$ Lax pair of Reiman–Semenov-T.-Shanski (1987).

"Small" $(2r \times 2r)$ matrix Lax representation

Generalization of the 2×2 Mumford Lax pair:

Theorem

Up to the action of the discrete group generated by reflections $(X, P) \longmapsto (\pm X, \pm P)$*, the Neumann flows with* T_κ *are equivalent to*

$$\frac{d}{dt} L(\lambda) = [L(\lambda), A_\kappa(\lambda)], \qquad \lambda \in \mathbb{C},$$

$$L(\lambda) = \begin{pmatrix} X^T(\lambda \mathbf{I}_n - A)^{-1} P & X^T(\lambda \mathbf{I}_n - A)^{-1} X \\ \mathbf{I}_r - P^T(\lambda \mathbf{I}_n - A)^{-1} P & -P^T(\lambda \mathbf{I}_n - A)^{-1} X \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \mathbf{I}_r & 0 \end{pmatrix} + \sum_i \frac{\mathcal{N}_i}{\lambda - a_i} \in sp(2r),$$

Note: The Lax matrix does not give the $so(r)$-momentum non-commutative integrals $\Psi_{ij} = (X^T P - P^T X)_{ij}$.

- *What are generic invariant tori ?*

Theorem (B. Jovanovic, Yu. F.)

If all the eigenvalues of A are distinct and rank $\Psi = [r/2]$ (maximal), then the Neumann systems are completely integrable in the non-commutative sense. The generic motions of the system are quasi-periodic over isotropic tori of dimension

$$\boxed{\delta = \frac{1}{2}\left(2r(n-r) + \frac{r(r-1)}{2} - \left[\frac{r}{2}\right]\right) + \left[\frac{r}{2}\right]}$$

$$\delta < \frac{1}{2}\,\mathrm{Dim}\, T^*V(n,r), \quad (r > 1)$$

If $\Psi = 0$, then the tori have dimension $r(n-r)$.

The spectral curve $\mathcal{S}$ of $L(\lambda)$ I

$$L(\lambda) = \begin{pmatrix} X^T(\lambda \mathbf{I}_n - A)^{-1}P & X^T(\lambda \mathbf{I}_n - A)^{-1}X \\ \mathbf{I}_r - P^T(\lambda \mathbf{I}_n - A)^{-1}P & -P^T(\lambda \mathbf{I}_n - A)^{-1}X \end{pmatrix}$$

$$\begin{aligned} \mathrm{F}(\lambda, w) &= |a(\lambda)L(\lambda) - w\mathbf{I}_{2r}| \\ &\equiv w^{2r} + w^{2r-2}a(\lambda)\mathcal{I}_2(\lambda) + \cdots \\ &\qquad + w^2 a^{2r-3}(\lambda)\mathcal{I}_{2r-2}(\lambda) + a^{2r-1}(\lambda)\mathcal{I}_{2r}(\lambda) = 0\,, \end{aligned}$$

$\mathcal{I}_{2l}(\lambda)$ being a polynomial of degree $n - l$.

- Over $\lambda = a_i$ the curve $\mathcal{S}$ has singularity $\delta_i = (2r-1)(r-1)$.
- Singularity at the infinite part with local coordinates

$$\lambda = \frac{1}{t}, \quad w = \frac{\mathfrak{w}}{t^n}$$

Then $F(\lambda, w) = 0 \Longrightarrow (\mathfrak{w}^2 - t)^r + o_r(\mathfrak{w}, t) = 0$,
and $\mathcal{S}$ has a strong singularity at ∞.

The spectral curve $\mathcal{S}$ of $L(\lambda)$ II

The eigenvector equation $a(\lambda)L(\lambda)\psi = w\psi$, $\psi \in \mathbb{P}^{2r-1}$ gives

$$\begin{pmatrix} \mathcal{V}_0 t + \mathcal{V}_1 t^2 + \cdots & \mathbf{I}_r t + \mathcal{U}_1 t^2 + \cdots \\ \mathbf{I}_r + \mathcal{W}_0 t + \mathcal{W}_1 t^2 + \cdots & -\mathcal{V}_0^T t - \mathcal{V}_1^T t^2 + \cdots \end{pmatrix} \psi = \mathfrak{w}(t)\psi,$$

$$\mathcal{V}_0 = X^T P, \quad \mathcal{V}_1 = -\operatorname{Tr} A\,\mathbf{I}_r + X^T AP,$$

$$\mathcal{U}_1 = -\operatorname{Tr} A\,\mathbf{I}_r + X^T AX, \quad \mathcal{W}_0 = -\operatorname{Tr} A\,\mathbf{I}_r - P^T P$$

and the Puiseaux expansions: If $\mathcal{V}_0 = \mathrm{X}^T P \neq 0$ and r is *impair*, then

$$\begin{aligned} \mathfrak{w}_s(t) &= t^{1/2} + f_s t + b_s t^{3/2} + \cdots, \\ \mathfrak{w}_{[r/2]+s}(t) &= t^{1/2} - f_s t + b_s t^{3/2} + \cdots, \qquad s = 1, \ldots, [r/2], \\ \mathfrak{w}_r(t) &= t^{1/2} \qquad\qquad + b_0 t^{3/2} + B_0 t^{5/2} + \cdots, \end{aligned}$$

f_s are *distinct*,

$$\psi = \begin{pmatrix} \mathbf{0} + O(t^{1/2}) \\ \mathbf{1}_s + O(t^{1/2}) \end{pmatrix}.$$

The spectral curve $\mathcal{S}$ of $L(\lambda)$ III

The order of singularity at ∞:

$$\delta_\infty = 2nr(nr - 2r - 1) + 2r(r+1) \qquad \text{if } \operatorname{rank}(\mathcal{V}_0) \text{ is maximal},$$
$$\delta_\infty = 2nr(nr - 2r - 1) + r^2 + \frac{3r(r+1)}{2} \qquad \text{if } \mathcal{V}_0 = 0.$$

Degree of $\mathcal{S}$ equals $N = n(2r-1) + (n-r)$, and the geometric genus

$$g = \frac{(N-1)(N-2)}{2} - \sum_{i=1}^{n} \delta_i - \delta_\infty$$
$$= 2nr - n - \frac{3}{2}r^2 - \frac{1}{2}r + 1 \qquad \text{if } \operatorname{rank}(\mathcal{V}_0) = 2[r/2],$$
$$g = 2r(n-r) - n + 1 \qquad \text{if } \mathcal{V}_0 = 0\,.$$

The regularized spectral curve $\mathcal{S}'$ in the case $r = 4$

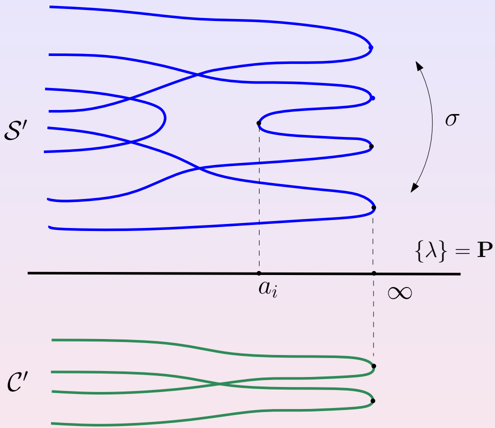

Figure: The 2-fold covering $\mathcal{S}' \to \mathcal{C}' = \mathcal{S}'/\sigma$ ramified at $2n - 2[r/2]$ points

σ extends to $\mathrm{Jac}(\mathcal{S}') = \mathbb{C}^g/\Lambda \cong \mathrm{Jac}(\mathcal{C}) \oplus \mathrm{Prym}(\mathcal{S}'/\sigma)$

Dimension of $\mathrm{Prym}(\mathcal{S}'/\sigma)$ is $l = \frac{1}{2}\left(2r(n-r) + \frac{r(r-1)}{2} - \left[\frac{r}{2}\right]\right)$.

Introduce $[r/2]$ meromorphic differentials Ω_i with pairs of simple poles at $\infty_i, \sigma(\infty_i)$ such that $\sigma^* \Omega_i = -\Omega_i$

Consider generalized Abel map

$$\tilde{\mathcal{A}}(P) = \int_{P_0}^{P} (\omega_1, \dots, \omega_g, \Omega_1, \dots, \Omega_{[r/2]})^T \in \mathbb{C}^{g+[r/2]}$$

And the generalized Jacobian

$$\widetilde{\mathrm{Jac}}(\mathcal{S}', \Omega_i) = \mathbb{C}^{g+[r/2]}/\tilde{\Lambda} \cong \mathrm{Jac}(\mathcal{S}') \times \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{[r/2]}.$$

σ extends also to $\widetilde{\mathrm{Jac}}(\mathcal{S}', \Omega_i) \cong \mathrm{Jac}(\mathcal{C}) \oplus \underbrace{\widetilde{\mathrm{Prym}}(\mathcal{S}'/\sigma, \Omega_i)}_{\text{generalized Prym variety}}$.

Note: $\dim \widetilde{\mathrm{Prym}}(\mathcal{S}'/\sigma, \Omega_i) = \delta$.

Theorem (B. Jovanovic, Yu. F.)

A generic complex δ-dimensional invariant manifold of the Neumann system on $V(n, r)$ is an open subset of $\widetilde{\mathrm{Prym}}(\mathcal{S}'/\sigma, \Omega_i)$.

Discretization of the Neumann systems on $V(n, r)$

A family of transformations $\mathcal{B}_{\lambda^*} : (X, P) \mapsto (\tilde{X}, \tilde{P})$, $\lambda^* \in \mathbb{C}$

$$
\begin{aligned}
P &= \ A^{1/2}(\lambda^*)\,\tilde{X} - X\,B(\lambda^*),\\
\tilde{P} &= -A^{1/2}(\lambda^*)\,X + \tilde{X}\,B(\lambda^*),\\
A(\lambda) &= \lambda \mathbf{I}_n - A,\\
B(\lambda^*) &= \frac{1}{2}\left(\tilde{X}^T A^{1/2}(\lambda^*) X + X^T A^{1/2}(\lambda^*)\tilde{X}\right) \in \mathrm{Symm}(r \times r).
\end{aligned}
$$

The alternative form (discrete Lagrange equations on $V(n, r)$)
($\lambda^* = 0$, Veselov–Moser, 1991)

$$
\boxed{X + \tilde{\tilde{X}} = A^{-1/2}(\lambda^*)\,\tilde{X}\,B},
$$

$$
B = \frac{1}{2}\left(\tilde{X}^T A^{1/2}(\lambda^*)(X + \tilde{\tilde{X}}) + (X + \tilde{\tilde{X}})^T A^{1/2}(\lambda^*)\tilde{X}\right)
$$

The matrix quadratic equation I

To evaluate B and $\mathcal{B}_{\lambda^*}$, we arrive at $r \times r$ matrix quadratic equation w.r.t. B,

$$\boxed{B\mathcal{U}B + B\mathcal{V} + \mathcal{V}^T B - \mathcal{W} = 0}$$

$$\mathcal{U} = X^T A^{-1}(\lambda^*)X, \quad \mathcal{V} = X^T A^{-1}(\lambda^*)P, \quad \mathcal{W} = \mathbf{I}_r - P^T A^{-1}(\lambda^*)P.$$

To solve it, introduce $\begin{pmatrix} \mathcal{V} & \mathcal{U} \\ \mathcal{W} & -\mathcal{V}^T \end{pmatrix} = L(\lambda^*)$.

Its eigenvalues $\{w_1, \dots, w_{2r}\}$ are divided into r pairs $(w_i, -w_i)$, $i = 1, \dots, r$.

$\Longrightarrow \quad \exists\ 2^r$ possible partitions $\{w_1, \dots, w_r \mid -w_1, \dots, -w_r\}$.

Let $\psi_1, \dots, \psi_r \in \mathbb{C}^{2r}$ be the eigenvectors of $L(\lambda^*)$ with *distinct* eigenvalues $w_1, \dots, w_r$ such that $w_i \neq -w_j$ and $\Psi = (\psi_1 \cdots \psi_r)$ be *a non-special eigenmatrix*.

The matrix quadratic equation II

Proposition (J. Potter, 1964)

Any <u>*symmetric*</u> *solution of the matrix quadratic equation*

$$B\mathcal{U}B + B\mathcal{V} + \mathcal{V}^T B - \mathcal{W} = 0$$

has the form

$$B = \Theta\, \Xi^{-1},$$

where Ξ, Θ *are upper and lower* $r \times r$ *halves of a non-special eigenmatrix* $\Psi = (\psi_1 \cdots \psi_r) = \begin{pmatrix} \Xi \\ \Theta \end{pmatrix}$ *of* $\begin{pmatrix} \mathcal{V} & \mathcal{U} \\ \mathcal{W} & -\mathcal{V}^T \end{pmatrix}$.

Corollary. For generic λ^*, the *complex* map $\mathcal{B}_{\lambda^*}$ is 2^r-valued.

- *Why the integrals of the continuous system are preserved ?*

Theorem

The discrete Neumann system on $V(n,r)$ is equivalent to the intertwining $2r \times 2r$ matrix relation

$$\tilde{L}(\lambda)M(\lambda|\lambda^*) = M(\lambda|\lambda^*)L(\lambda),$$

$$L(\lambda) = \begin{pmatrix} X^T(\lambda \mathbf{I}_n - A)^{-1}P & X^T(\lambda \mathbf{I}_n - A)^{-1}X \\ \mathbf{I}_r - P^T(\lambda \mathbf{I}_n - A)^{-1}P & -P^T(\lambda \mathbf{I}_n - A)^{-1}X \end{pmatrix}$$

(as in the continuous case),

$$M(\lambda|\lambda^*) = \begin{pmatrix} -B(\lambda^*) & \mathbf{I}_r \\ (\lambda - \lambda^*)\mathbf{I}_r + B^2(\lambda^*) & -B(\lambda^*) \end{pmatrix},$$

Description of $\mathcal{B}_{\lambda^*}$ on the complex invariant tori (case $X^T P = 0$)

Theorem

The 2^r*-valued map* $\mathcal{B}_{\lambda^*}$ *is given by translations by one of the following* 2^r *vectors in* $\operatorname{Prym}(\mathcal{S}'/\sigma)$

$$T = \mathcal{A}(\underbrace{\lambda^*, w_1}_{Q_1}) + \cdots + \mathcal{A}(\underbrace{\lambda^*, w_r}_{Q_r}) - \mathcal{A}(\infty_1) - \cdots - \mathcal{A}(\infty_r),$$

$$\mathcal{A}(P) = \int_{P_0}^{P} (\omega_1, \ldots, \omega_g)^T \in \mathbb{C}^g$$

Compare with the translation on $\operatorname{Jac}(\Gamma)$ in the classical case $r = 1$:

$$T = \mathcal{A}(\mathcal{P}) - \mathcal{A}(\infty), \quad \mathcal{P} = (\lambda^*, \mu^*)$$

.

Note :

$\sum Q_s - \sum \infty_s + \sigma(\sum Q_s - \sum \infty_s) = (\lambda - \lambda^*)$, so $T + \sigma T = 0$.

Theorem

The 2^r-valued map $\mathcal{B}_{\lambda^*}$ is given by translations by one of the following vectors

$$T = \mathcal{A}(Q_1) + \cdots + \mathcal{A}(Q_r) - \mathcal{A}(\infty_1) - \cdots - \mathcal{A}(\infty_r),$$

Sketch of proof. $L(\lambda)\psi = w\psi$, $\psi(P) = (\psi^1(P), \ldots, \psi^{2r}(P))^T$, $P \in \mathcal{S}'$.

$$\tilde{L}(\lambda)M(\lambda|\lambda^*) = M(\lambda|\lambda^*)L(\lambda) \Longrightarrow$$

$$\tilde{\psi}(P) = M(\lambda, \lambda^*)\psi(P) = \begin{pmatrix} -\Gamma^* & \mathbf{I}_r \\ (\lambda - \lambda^*)\mathbf{I}_r + (\Gamma^*)^2 & -\Gamma^* \end{pmatrix} \psi(P)$$

$\tilde{(}\psi_i) > -\tilde{\mathcal{D}}$

Note: $\det M(\lambda, \lambda^*) = (\lambda - \lambda^*)^r$.

$$\tilde{\mathcal{D}} + Q_1 + \cdots + Q_r = \mathcal{D} + \infty_1 + \cdots + \infty_r.$$