Elliptic formal group laws and differential equations

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[1] Victor M. Buchstaber, Elena Yu. Bunkova
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[2] Victor M. Buchstaber "The general Krichever genus"
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Starting with the general elliptic curve

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$$

we will find the formal group law as a rational function in terms of the Tate coordinates.

We will construct a function $\Psi(u; v, \alpha)$ - the solution of the equation (the generalized Lame equation)

$$\Psi''(u) - \left(2\wp(u) - \frac{1}{4}\frac{\wp'(v)^2 - \wp'(w)^2}{(\wp(u) - \wp(v))^2}\right)\Psi(u) = \wp(v)\Psi(u)$$

with a periodic potential:

$$\Psi(u) = \frac{\sigma(u+v)^{\frac{1}{2}(1-\alpha)}\sigma(v-u)^{\frac{1}{2}(1+\alpha)}}{\sigma(u)\sigma(v)} \exp\left(\alpha\zeta(v)u\right), \qquad \alpha = \frac{\wp'(w)}{\wp'(v)}.$$

We will introduce the general elliptic Hirzebruch genus.
This genus is defined by a 5-parametric family of elliptic functions (of order 3 in the general case).
It takes $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$-integer values on any stably complex manifold.

The general elliptic curve in $\mathbb{C}P^2$ in Weierstrass form

$$Y^2Z + \mu_1 XYZ + \mu_3 YZ^2 = X^3 + \mu_2 X^2 Z + \mu_4 XZ^2 + \mu_6 Z^3.$$

The Weierstrass parametrization. In the coordinate map $Z \neq 0$ with the coordinates $x = X/Z$, $y = Y/Z$ we have

$$y^2 + \mu_1 xy + \mu_3 y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

The Tate parametrization. In the coordinate map $Y \neq 0$ with the coordinates $t = -X/Y$, $s = -Z/Y$ we have

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3.$$

In the map $X \neq 0$ with the coordinates $v = Y/X$ and $w = Z/X$

$$vw(v + \mu_1 + \mu_3 w) = 1 + \mu_2 w + \mu_4 w^2 + \mu_6 w^3.$$

The degrees $\deg X = -4$, $\deg Y = -6$, $\deg Z = 0$, $\deg \mu_i = -2i$.

One can pass to the **standard Weierstrass form**

$$y^2 = 4x^3 - g_2 x - g_3$$

by the affine transform $x \to x + \frac{1}{12}(4\mu_2 + \mu_1^2)$, $y \to 2y + \mu_1 x + \mu_3$.

Then

$$g_2 = \frac{1}{12}(4\mu_2 + \mu_1^2)^2 - 2(\mu_1\mu_3 + 2\mu_4),$$
$$g_3 = \frac{1}{6}(4\mu_2 + \mu_1^2)(\mu_1\mu_3 + 2\mu_4) - \frac{1}{6^3}(4\mu_2 + \mu_1^2)^3 - 4\mu_6 - \mu_3^2.$$

The Weierstrass uniformization

$$\wp(u)'^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

Tate coordinates.

$$s(t) = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3.$$

$s(t) \in \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6][[t]]$.

Let $\mu(v) = (0, 3v + c_2, c_3, 3v^2 + 2c_2 v + c_4, v^3 + c_2 v^2 + c_4 v + c_6)$.
The function $S(t, v) = s(t, \mu(v))$ satisfies **the Hopf equation**

$$\frac{\partial S}{\partial v} = S \frac{\partial S}{\partial t}.$$

The path $\mu(v)$ defines a family of elliptic curves with the same standard Weierstrass form for any v.

The formal group law. A commutative one-dimensional formal group law over the ring A is a formal series

$$F(t_1, t_2) = t_1 + t_2 + \sum_{i,j} \alpha_{i,j} t_1^i t_2^j, \ \ \alpha_{i,j} \in A,$$

such that the following conditions hold:

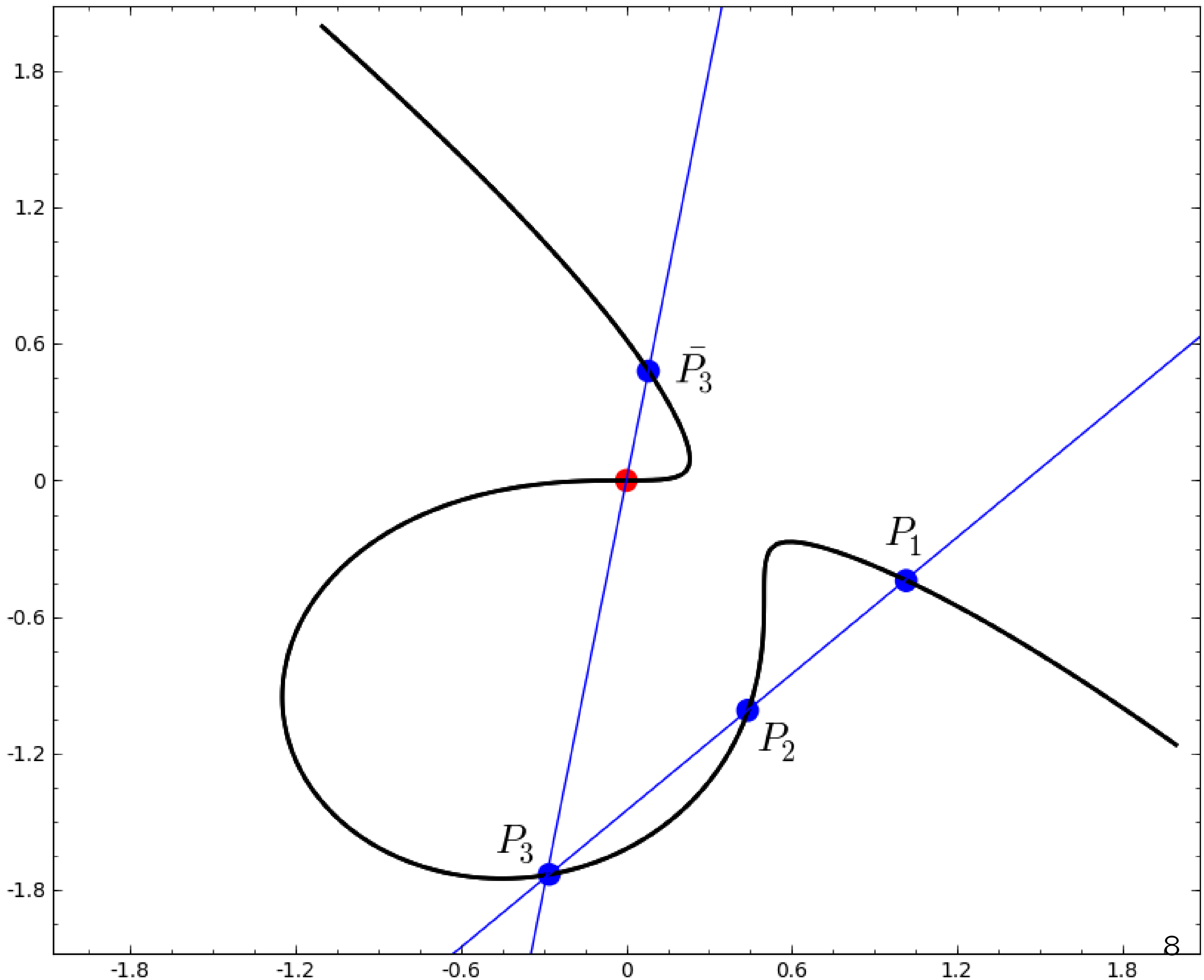
$$F(t, 0) = t, \quad F(t_1, t_2) = F(t_2, t_1), \quad F(t_1, F(t_2, t_3)) = F(F(t_1, t_2), t_3).$$

The exponential of the formal group law. For each formal group law $F \in A[[t_1, t_2]]$ there exists a series $f(u) \in A \otimes \mathbb{Q}[[u]]$, such that $f(0) = 0$, $f'(0) = 1$, and

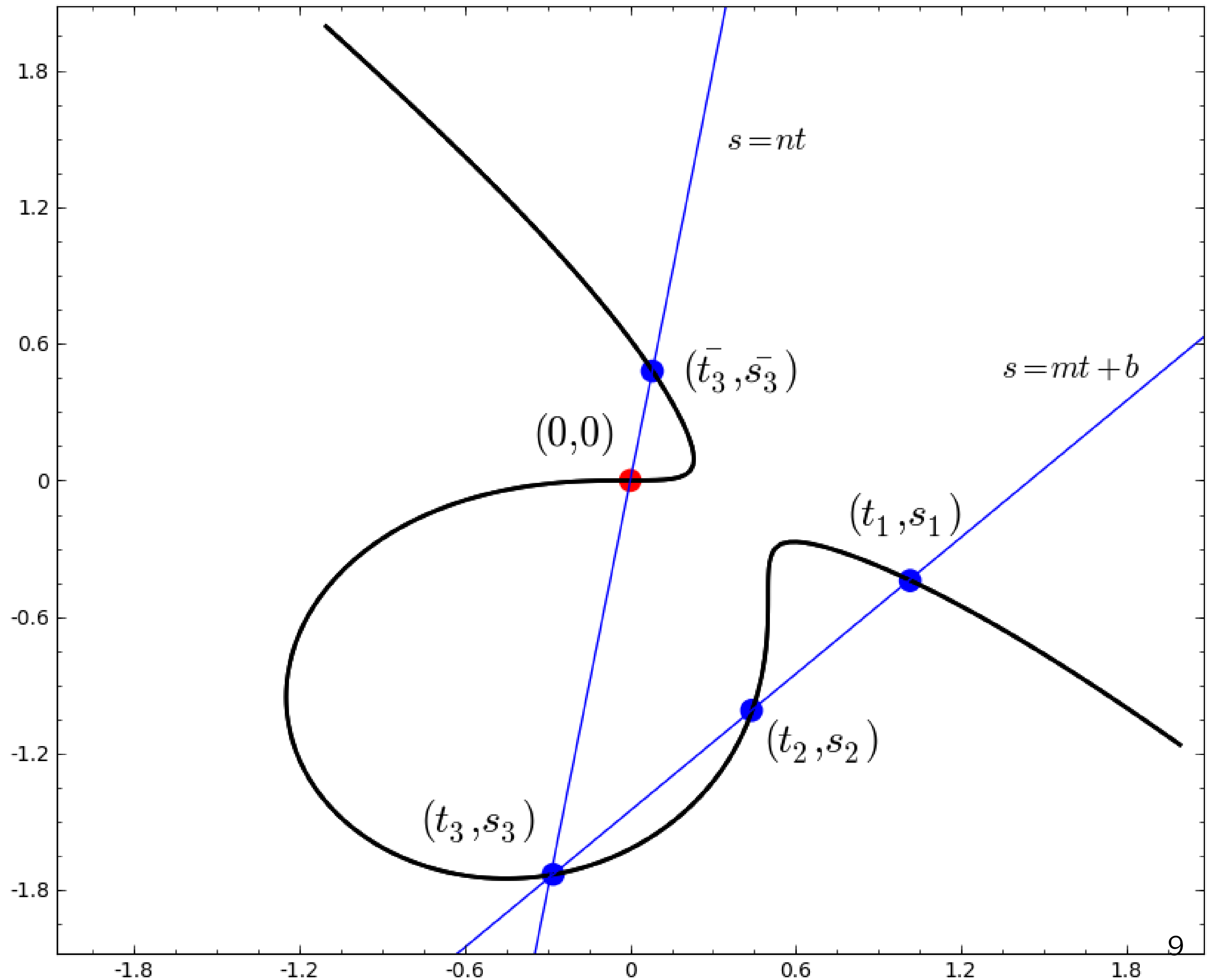
$$f(u + v) = F(f(u), f(v)).$$

We will need the formula

$$f'(u) = \frac{\partial}{\partial t_2} F(f(u), t_2)\Big|_{t_2=0}.$$



1.8
1.2
0.6
0
-0.6
-1.2
-1.8
-1.8
-1.2
-0.6
0
0.6
1.2
1.8
$\bar{P_3}$
P_1
P_2
P_3



$s = nt$
$s = mt + b$
$(\bar{t_3}, \bar{s_3})$
$(0,0)$
(t_1, s_1)
(t_2, s_2)
(t_3, s_3)
1.8
1.2
0.6
0
-0.6
-1.2
-1.8
-1.8
-1.2
-0.6
0
0.6
1.2
1.8

$$s = t^3 + \mu_1 ts + \mu_2 t^2 s + \mu_3 s^2 + \mu_4 ts^2 + \mu_6 s^3.$$

$$s = nt + b, \qquad s = nt.$$

$$m = \frac{s_1 - s_2}{t_1 - t_2}, \qquad b = \frac{t_1 s_2 - t_2 s_1}{t_1 - t_2}, \qquad n(t_1, t_2) = m + \frac{b}{t_3}.$$

We have $\bar{t} = -t$ if and only if $\mu_1 = \mu_3 = 0$.

Theorem 1. The general elliptic formal group law $F_\mu(t_1, t_2)$ is given by the formula

$$F_\mu(t_1,t_2) = ((t_1+t_2)(1-\mu_3 b-\mu_6 b^2)-t_1 t_2(\mu_1+\mu_3 m+\mu_4 b+2\mu_6 bm))\times \\ \times \frac{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}{(1+\mu_2 n+\mu_4 n^2+\mu_6 n^3)(1-\mu_3 b-\mu_6 b^2)^2}.$$

Corollary. $F_\mu(t_1,t_2) \in \mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_4,\mu_6][[t_1,t_2]]$.

Example. In the case $(\mu_1,\mu_3) = (0,0)$ we get

$$F_\mu(t_1,t_2) = t_1 + t_2 + b\frac{(\mu_2 + 2\mu_4 m + 3\mu_6 m^2)}{(1+\mu_2 m+\mu_4 m^2+\mu_6 m^3)}.$$

Example. In the case $(\mu_3, \mu_4, \mu_6) = (0, 0, 0)$ we get

$$F(t_1, t_2) = \frac{t_1 + t_2 - \mu_1 t_1 t_2}{1 + \mu_2 t_1 t_2}, \quad \text{where } \mu_1 = \alpha + \beta,\ \mu_2 = -\alpha\beta.$$

This formal group law corresponds to the remarkable two-parameter Todd genus $T_{\alpha,\beta} \colon \Omega_U \to \mathbb{Z}[\mu_1, \mu_2]$, a particular case of which are the famous Hirzebruch genera: the Todd genus $(\mu_2 = 0)$; the signature $(\mu_1 = 0)$; the Euler characteristic $(\mu_1^2 = -4\mu_2)$.

Example. In the case $(\mu_4, \mu_6) = (0, 0)$ we get

$$F_\mu(t_1, t_2) = \frac{(t_1 + t_2)(1 - \mu_3 b) - \mu_1 t_1 t_2 - \mu_3 t_1 t_2 m}{(1 - \mu_3 b)(1 + \mu_2 t_1 t_2 - \mu_3 b)}.$$

The Hizrebruch genus.

$$\text{Let} \quad f(u) = u + \sum_{k\ge 1} f_k u^{k+1}, \quad \text{where} \quad f_k \in A \otimes \mathbb{Q}.$$

$$\text{Define} \quad \prod_{i=1}^{n} \frac{u_i}{f(u_i)} = L_f(\sigma_1, \ldots, \sigma_n),$$

where σ_k is the k-th elementary symmetric polynomial of $u_1, \ldots, u_n$.

The Hirzebruch genus $L_f(M^{2n})$ *of a stably complex manifold* M^{2n} is the value of the cohomology class $L_f(c_1, \ldots, c_n)$ on the fundamental cycle $\langle M^{2n} \rangle$ of the manifold M^{2n}, where c_k is the k-th Chern class of the tangent bundle of the manifold M^{2n}.

The Hirzebruch genus L_f is called A-*integer* if $L_f(M^{2n}) \in A_{-2n}$ for any stably complex manifold M^{2n}.

Let us take a formal group $F(t_1, t_2)$ over the ring A and its exponential $f(u)$. Then the corresponding Hirzebruch genus L_f is A-integer.

Corollary. (The general elliptic genus.)
Theorem 1 gives a 5-parametric $\mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]$-integer Hirzebrugh genus.

$$\frac{\partial}{\partial t_2} F_\mu(t, t_2)\Big|_{t_2=0} = 1 - \mu_1 t - \mu_2 t^2 - 2\mu_3 s - 2\mu_4 ts - 3\mu_6 s^2.$$

Let $\phi(u) = s(f(u))$. We obtain

$$\begin{cases} f' = 1 - \mu_1 f - \mu_2 f^2 - 2\mu_3\phi - 2\mu_4 f\phi - 3\mu_6\phi^2 \\ \phi = f^3 + \mu_1 f\phi(u) + \mu_2 f^2\phi + \mu_3\phi^2 + \mu_4 f\phi^2 + \mu_6\phi^3. \end{cases}$$

Theorem 2. Let $(\mu_3, \mu_4, \mu_6) \neq (0, 0, 0)$. Then the exponential $f(u)$ of the elliptic formal group law F_μ is the solution of the equation

$$\mu_6 f'^3 + P_2(f) f'^2 = P_6(f),$$

$$P_2(f) = (\mu_4^2 - 3\mu_2\mu_6)f^2 + (2\mu_3\mu_4 - 3\mu_1\mu_6)f + (3\mu_6 + \mu_3^2), \quad P_6(f) = \ldots$$

with the initial condition $f(0) = 0$ and the condition $f'(0) = 1$ which fixes the branch of solutions.

Let $(\mu_3, \mu_4, \mu_6) = (0, 0, 0)$. Then the exponential $f(u)$ of the elliptic formal group law F_μ is the solution of the equation

$$f' = 1 - \mu_1 f - \mu_2 f^2$$

with the initial condition $f(0) = 0$.

Example. Let $\mu_6 = 0$, $(\mu_3, \mu_4) \neq (0, 0)$:

$$f'^2 = 1 - 2\mu_1 f + (\mu_1^2 - 2\mu_2) f^2 + (2\mu_1\mu_2 - 4\mu_3) f^3 + (\mu_2^2 - 4\mu_4) f^4.$$

Example. Let $(\mu_1, \mu_2, \mu_3, \mu_4) = (0, 0, 0, 0)$:

$$(f' - 1)(f' + 2)^2 = -27\mu_6 f^6.$$

The exponential of the general elliptic formal group is

$$f(u) = -2\frac{\wp(u; g_2, g_3) - \frac{1}{12}(4\mu_2 + \mu_1^2)}{\wp'(u; g_2, g_3) - \mu_1\wp(u; g_2, g_3) + \frac{1}{12}\mu_1(4\mu_2 + \mu_1^2) - \mu_3}.$$

We have

$$-\frac{1}{f(u)} = \frac{1}{2}\frac{\wp'(u) + \wp'(w)}{\wp(u) - \wp(v)} - \frac{\mu_1}{2},$$

where $\wp(u) = \wp(u; g_2, g_3)$, $\wp'(w) = -\mu_3$ and $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$.

The exponential of the elliptic formal group law in the non-degenerate case is the ellipitc function of order 2 iff $\mu_6 = 0$.
It is the ellipitc function of order 3 in the general non-degenerate case.

The Baker-Akhiezer function.

$$\Phi(u;v) = \frac{\sigma(v-u)}{\sigma(u)\sigma(v)} \exp(\zeta(v)u).$$

The Lame equation.

$$\Phi''(u) - 2\wp(u)\Phi(u) = \wp(v)\Phi(u).$$

The quasiperiodic solutions such that $\lim_{u\to 0}\left(\Phi(u) - \frac{1}{u}\right) = 0$ are $\Phi(u;v)$ and $\Phi(u;-v)$.

The periodic properties are

$$\Phi(u+2\omega_k;v) = \Phi(u;v)\exp(2\zeta(v)\omega_k - 2\eta_k v),$$
$$\Phi(u;v+2\omega_k) = \Phi(u;v).$$

Thus $(\ln\Phi(u))'$ is a doubly periodic meromorphic function of u:

$$(\ln\Phi(u))' = \zeta(u-v) + \zeta(v) - \zeta(u) = \frac{1}{2}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}.$$

The Krichever genus.
Let $f_0(u) = \frac{1}{\Phi(u)}$, where $\Phi(u)$ is the Baker-Akhiezer function.

I. M. Krichever showed L_{f_0} obtains the property of rigidity on SU-manifolds (Calabi–Yau manifolds) with the action of a circle.

The Hirzebruch genus defined by the series $f_{Kr}(u) = f_0(u)\exp(a_1 u)$, is called *the Krichever genus*.

Consider the transform

$$T\Big(\phi(u)\Big) = \frac{\phi(u)}{\phi'(u)}.$$

Consider the function $\widehat{\Phi}(u;v) = \Phi(u;v)\exp(-\frac{\mu_1}{2}u)$, where $\Phi(u;v)$ is the Baker-Akhiezer function, and the function $U_1(u) = U_1(u;v) = -\frac{1}{2}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}$.

The function $\widehat{\Phi}(u) = \widehat{\Phi}(u;v)$ is a solution of the equation

$$\widehat{\Phi}''(u) - (2\wp(u) + \mu_1 U_1(u))\widehat{\Phi}(u) = (\wp(v) + \frac{\mu_1^2}{4})\widehat{\Phi}(u).$$

Corollary.

Let $f_\mu(u)$ be the exponential of the formal group law $F_\mu(t_1, t_2)$ where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, 0)$. Then

$$(\ln \widehat{\Phi}(u;v))' = -\frac{1}{f_\mu(u)},$$

where v is determined by $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$ and $\wp'(v) = -\mu_3$.

In the case of general μ, let

$$\Psi(u) = \frac{\sigma(u+v)^{\frac{1}{2}(1-\alpha)}\sigma(v-u)^{\frac{1}{2}(1+\alpha)}}{\sigma(u)\sigma(v)} \exp\left((-\frac{\mu_1}{2} + \alpha\zeta(v))u\right),$$

where $\sigma(u) = \sigma(u; g(\mu))$. Then

$$(\ln \Psi(u))' = \frac{1}{2}\frac{\wp'(u) + \alpha\wp'(v)}{\wp(u) - \wp(v)} - \frac{\mu_1}{2}.$$

Corollary. Let $f_\mu(u)$ be the exponential of the general elliptic formal group law. Then

$$(\ln \Psi(u))' = -\frac{1}{f_\mu(u)}$$

for $\wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2)$, $\alpha = \frac{\wp'(w)}{\wp'(v)} = \frac{-\mu_3}{\sqrt{\mu_3^2+4\mu_6}}$.

The function $\Psi(u)$ with $\alpha = \frac{\wp'(w)}{\wp'(v)}$ satisfies the equation

$$\Psi''(u) - (2\wp(u) + \mu_1 U_1(u) - U_2(u))\Psi(u) = (\wp(v) + \frac{\mu_1^2}{4})\Psi(u)$$

where $U_2(u) = U_2(u; v, w) = U_1(v; u, w)U_1(v; u, -w)$ and

$$U_1(u; v, w) = -\frac{1}{2}\frac{\wp'(u) + \wp'(w)}{\wp(u) - \wp(v)}.$$

We have

$$\Psi(u) = \Phi(u; v) \exp(-\frac{\mu_1}{2}u) \left(\frac{\Phi(u; -v)}{\Phi(u; v)}\right)^{\frac{1}{2}(1-\alpha)} .$$

The periodic properties are

$$\Psi(u + 2\omega_k; v) = \Psi(u; v) \exp(\alpha(2\zeta(v)\omega_k - 2\eta_k v) - \mu_1 \omega_k).$$

Example. $(\mu_1 = 0)$.

$$\Psi''(u) - (2\wp(u) - \frac{1}{4}\frac{\wp'(v)^2 - \wp'(w)^2}{(\wp(u) - \wp(v))^2})\Psi(u) = \wp(v)\Psi(u)$$

Example. $(\mu_1 = 0,\ \mu_3 = 0 \Rightarrow \alpha = 0)$.

$$\Psi(u) = \sqrt{\Phi(u;v)\Phi(u;-v)} = \frac{\sqrt{\sigma(u+v)\sigma(v-u)}}{\sigma(u)\sigma(v)} = \sqrt{\wp(u) - \wp(v)},$$

$\wp(v) = \frac{1}{3}\mu_2$, $(\wp(v)')^2 = 4\mu_6$.

$$\Psi(u)^2\Psi'(u)^2 = \Psi(u)^6 + \mu_2\Psi(u)^4 + \mu_4\Psi(u)^2 + \mu_6.$$

In the case $\mu_6 = 0$ we come to

$$\Psi'(u)^2 = \Psi(u)^4 + \mu_2\Psi(u)^2 + \mu_4.$$

A **Hurwitz series** over A is a formal power series in the form

$$\varphi(u) = \sum_{k \geqslant 0} \varphi_k \frac{u^k}{k!} \ \in \ A \otimes \mathbb{Q}[[u]]$$

with $\varphi_k \in A$ for all $k = 0, 1, 2, 3, \ldots$.

For the sigma function

$$\sigma(u) = u \sum_{i,j \geq 0} \frac{a_{i,j}}{(4i+6j+1)!} (\frac{g_2 u^4}{2})^i (2 g_3 u^6)^j,$$

there is the Weierstrass recursion

$$a_{i,j} = 3(i+1) a_{i+1,j-1} + \\ + \frac{16(j+1)}{3} a_{i-2,j+1} - \frac{(4i+6j-1)(2i+3j-1)}{3} a_{i-1,j}$$

$$a_{0,0} = 1, \quad a_{i,j} = 0 \text{ for } i < 0 \text{ or } j < 0.$$

The sigma function is a Hurwitz series over $\mathbb{Z}[\frac{g_2}{2}, 2g_3]$:

$$\sigma(u) \in H\mathbb{Z}[\frac{g_2}{2}, 2g_3][[u]],$$

that is $a_{i,j} \in \mathbb{Z}$.

Corollary. For any v let $a_2 = \wp(v)$, $a_3 = \wp'(v)$, $a_4 = \frac{g_2}{2}$. Then $g_3 = -a_3^2 + 4a_2^3 - 2a_2a_4$. Thus $\sigma(u) \in H\mathbb{Z}[a_2, a_3, a_4][[u]]$.

Conjecture. Let $a(i,j) = 2^k 3^l s(i,j)$, where $s(i,j) \in \mathbb{Z}$ is coprime with 2 and 3. Let

$$\frac{(4i+6j+1)!}{2^{3i+4j}\, 3^{i+j}\, i!\, j!} = 2^{k_1} 3^{l_1} s_1(i,j),$$

where $s_1(i,j) \in \mathbb{Z}$ is coprime with 2 and 3.

Then $k = k_1$, $l = l_1$.

In the vicinity of $u = 0$ the exponential of the general Krichever genus

$$\frac{1}{\Psi(u)} = u + \sum \Psi_k \frac{u^{k+1}}{(k+1)!},$$

is a Hurwitz series over $\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, where

$$\begin{aligned}
a_1 &= \frac{\mu_1}{2}, \\
a_2 &= \wp(v) = \frac{1}{12}(4\mu_2 + \mu_1^2), \\
a_3 &= \wp'(w) = -\mu_3, \\
a_4 &= \frac{1}{2} g_2(\mu), \\
a_6 &= 4\mu_6 + \mu_3^2.
\end{aligned}$$