QUADRATIC IDENTITIES IN $\wp$-FUNCTIONS FOR HYPERELLIPTIC CURVES

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1

The tradition to which this work belongs is that of: Buchstaber, Enolskii & Leykin [3]; Cassels & Flynn [5]; Eilbeck, England, Matsutani, Onishi & Previato [6, 7]; Gibbons & Baldwin [4, 8]; Baker [2].

The author's work has appeared most recently in [1].

The motivation is to use simple representation theory to:

- understand the structure of the identities satisfied by the $\wp$-functions on Jacobians of general plane curves;
- to provide tools which significantly reduce the computational costs of obtaining such identities.

We will discuss hyperelliptic curves specifically under the following headings:

- transformation theory;
- the hyperelliptic curves of genus one, two and three;
- a conjecture for hyperelliptic curves of arbitrary genus.

2

C - a nonsingular curve of genus g over $\mathbb{C}$. Riemann-Roch tells us the dimensions of the divisor spaces, $L(D)$, of divisors D on C,

$$L(0) \subseteq L(D) \subseteq L(2D) \subseteq L(3D) \dots$$

with equality at the Weierstraß gaps for special divisors.

$x \in L(nD)$, $y \in L(mD) \Rightarrow xy \in L((n+m)D)$. For some least p there exists at least one linear relation, say $\phi = 0$, amongst the elements we so construct because the dimension of $L(pD)$ is restricted by Riemann-Roch.

There are $\mathbb{C}$-linear maps between the $L(nD)$:

$$\Lambda_i : L(nD) \to L((n+i)D).$$

In particular (in the case that the linear relation $\phi = 0$ is unique)

$$\Lambda_{-1}\phi = 0, \quad \Lambda_0\phi = c\phi, \quad \Lambda_1\phi = z\phi$$

for some $c \in L(0)$ and $z \in L(D)$.

2.1. **g=1.**

- $D = P$, a single point. $x \in L(2D)$, $y \in L(3D)$, $\phi = y^2 + [x]y + [x^3] \in L(6D)$. (*Here we use the notation* $[x^n]$ *to denote an arbitrary polynomial of degree* n *in* x.)
- $D = Q + R$. $x \in L(D)$, $y \in L(2D)$, $\phi = y^2 + [x^2]y + [x^4] \in L(4D)$.
- $D = Q + R + S$. $x, y \in L(D), \phi = y^3 + [x]y^2 + [x^2]y + [x^3] \in L(3D)$.

Each of these is a model of the genus one curve. (*The second is singular.*) There are maps between the models and linear maps between the $L(nD)$. (*For example, in the second instance,* $\partial_x : L(nD) \to L((n-1)D); x\partial_x,\ y\partial_y : L(nD) \to L(nD); x^2\partial_x : L(nD) \to L((n+1)D)$.)

2.2. **g=2.**

- $D = P$, a Weierstraß point. $L(0) = L(D)$, $L(2D) = L(3D)$, $x \in L(2D)$, $y \in L(5D), \phi = y^2 + [x^2]y + [x^5] \in L(10D)$.
- $D = Q + R$, a special divisor. $x \in L(D)$, $y \in L(3D), \phi = y^2 + [x^3]y + [x^6] \in L(6D)$.

3

In what follows we are interested in the general case of the hyperelliptic curve of genus g and its (*singular*) model,

$$y^2 = \sum_0^{2g+2} \binom{2g+2}{i} a_i x^i.$$

There exist amongst all the Λ_i the following three, particular maps

$$\begin{aligned} \Lambda_{-1} &= \partial_x - \sum_0^{2g+1} (2g+2-i) a_{i+1} \partial_{a_i} \\ \Lambda_0 &= -2x\partial_x - (g+1)y\partial_y - \sum_0^{2g+2} (2g+2-2i) a_i \partial_{a_i} \\ \Lambda_1 &= -x^2\partial_x - (g+1)xy\partial_y - \sum_1^{2g+2} i a_{i-1} \partial_{a_i} \end{aligned}$$

One may verify these satisfy $\mathfrak{sl}_2(\mathbb{C})$ commutation relations: $[\mathbf{e}, \mathbf{f}] = \mathbf{h}$, $[\mathbf{h}, \mathbf{e}] = 2\mathbf{e}$, $[\mathbf{h}, \mathbf{f}] = -2\mathbf{f}$. So it is appropriate to relabel them

$$\Lambda_{-1} \equiv \mathbf{e}, \quad \Lambda_0 \equiv \mathbf{h}, \quad \Lambda_1 \equiv \mathbf{f}.$$

The eigenvalue of the operator $\mathbf{h}$ applied to any monomial in x, y and the $a_i's$, assigns to it a *weight*, which may be negative. These weights are, in this approach, logically equivalent to the Sato weights in the standard approach.

One also sees that these linear operators are the infinitesimal form of the rational map:

$$x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta}, \quad y \mapsto \frac{y}{(\gamma x + \delta)^{g+1}}.$$

4

A (quite) general curve of genus g may be written

$$\phi = y^n + [x^p]y^{n-1} + [x^{2p}]y^{n-2} + \ldots + [x^{np}] = 0.$$

Under the transformations

$$\begin{aligned} x &\mapsto \frac{\alpha x + \beta}{\gamma x + \delta} \\ y &\mapsto \frac{y}{(\gamma x + \delta)^p} \end{aligned}$$

the holomorphic differentials, H, transform as a g-dimensional $\mathfrak{sl}_2(\mathbb{C})$ module which is not in general irreducible. One may verify by analysis of the transformation theory of 1-forms of the kind $\frac{x^2 i y^j dx}{\phi_{,y}}$ that in this (quite) general case

$$H = \bigoplus_{i=1}^{g} H_i$$

is the decomposition into irreducibles and

$$\dim H = \bigoplus_{i=1}^{g} \dim H_i.$$

Thus the cotangent space to the Jacobian variety of the curve has some interesting structure. This is simplest in the hyperelliptic case where H is isomorphic to the standard irreducible V_g of dimension g. The r index $\wp$-functions which we write for conciseness as $(ij) \equiv \wp_{ij}$, $(ijk) \equiv \wp_{ijk}$ etc. belong to the (not irreducible in general) modules, $Sym^r V_g^*$. (V^* *is the dual to* V. $V \cong V^*$.)

Our starting point is the Buchstaber et al. formulation of the inverse problem, deriving from Baker and ultimately Klein.

It is however first necessary to "tweak" the definition of the $\wp$-functions covariantly, that is in such a way that all objects are (co)variant but without altering their analytic properties. We deal with the genus one, two and three hyperelliptic curves.

In what follows $\wp$ is a function at a point $\mathbf{u} + \mathbf{u}_1 + \ldots + \mathbf{u}_g$ on the Jacobian, this point being the image under the Abel map of the divisor $(x_1, -y_1) + (x_2, -y_2) + \ldots + (x_g, -y_g)$.

5. Quadratics in genus one

$$(111)^2 = -\frac{1}{4} \det H$$

is the equation for the Weierstraß $\wp$ function (*equivalent to* (11) *in our present notation*), where

$$H = \begin{bmatrix} a_0 & 2a_1 & a_2 - \mathbf{2(11)} \\ 2a_1 & 4a_2 + \mathbf{4(11)} & 2a_3 \\ a_2 - \mathbf{2(11)} & 2a_3 & a_4 \end{bmatrix}$$

The curve:

$$y^2 = \mathbf{x} H \mathbf{x}^t = \sum_0^4 \binom{4}{i} a_i x^i$$

where $\mathbf{x} = (1, x, x^2)$.

The definition of $(ij) = (11)$, for (x_1, y_1) an arbitrary point on the curve:

$$\begin{aligned} yy_1 &= \mathbf{x}H\mathbf{x}_1^t \\ &= a_4x^2x_1^2 + 2a_3xx_1(x+x_1) + a_2(x^2+4xx_1x_1^2) + 2a_2(x+x_1) + a_0 \\ &= -2(x-x_1)^2(11) \end{aligned}$$

Rearranging

$$(11) = \frac{1}{2}\frac{a(x,x_1) - yy_1}{(x-x_1)^2}$$

where $a(x, x_1) - yy_1$ is the *covariant polar form* of the curve. The polar form can be thought of as a function on the curve with a double zero at the point (x_1, y_1) on the curve. Consequently (11) is regular at that point but has a second order pole at $(x_1, -y_1)$.

(11) is invariant under $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{f}$.

6. Quadratics in genus two

$$A(\mathbf{l})A(\mathbf{k}) = -\frac{1}{4}\det\begin{bmatrix} H & \mathbf{k}^t \\ \mathbf{l} & 0 \end{bmatrix}$$

are equations quadratic in the (ijk) where

$$A(\mathbf{l}) = [(222), -(122), (112), -(111)] \cdot \mathbf{l}^t$$

etc., and

$$H = \begin{bmatrix} a_0 & 3a_1 & 3a_2 - \mathbf{2(11)} & a_3 - \mathbf{2(11)} \\ 3a_1 & 9a_2 + \mathbf{4(11)} & 9a_3 + \mathbf{2(12)} & 3a_4 - \mathbf{2(22)} \\ 3a_2 - \mathbf{2(11)} & 9a_3 + \mathbf{2(12)} & 9a_4 + \mathbf{4(22)} & 3a_5 \\ a_3 - \mathbf{2(12)} & 3a_4 - \mathbf{2(22)} & 3a_5 & a_6 \end{bmatrix}$$

The curve:

$$y^2 = \mathbf{x}H\mathbf{x}^t = \sum_0^6 \binom{6}{i} a_i x^i$$

where $\mathbf{x} = (1, x, x^2, x^3)$.

The definition of the (ij), for (x_1, y_1) and (x_2, y_2) two arbitrary points on the curve:

$$yy_i = \mathbf{x}H\mathbf{x}_i^t \quad i = 1, 2$$

(6.1)

Rearranging

$$\frac{(11) + (x+x_i)(12) + xx_i(22)}{x - x_1} = \frac{1}{2}\frac{a(x,x_1) - yy_1}{(x-x_1)^3}$$

where $a(x, x_1) - yy_1$ is the *covariant polar form* of the sextic curve.

Each side of the above equation is invariant under $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{f}$. In particular

$$(11) \xrightarrow{\mathbf{e}} -2(12) \xrightarrow{\mathbf{e}/2} (22) \xrightarrow{\mathbf{e}} 0$$

$$0 \xleftarrow{\mathbf{f}} (11) \xleftarrow{\mathbf{f}/2} -2(12) \xleftarrow{\mathbf{f}} (22)$$

7. Quadratics in genus three

$$A(\mathbf{l},\mathbf{l}')A(\mathbf{k},\mathbf{k}') = -\frac{1}{4}\det\begin{bmatrix} H & \mathbf{k}^t & \mathbf{k}'^t \\ \mathbf{l} & 0 & 0 \\ \mathbf{l} & 0 & 0 \end{bmatrix}$$

are equations quadratic in the (ijk) where

$$A(\mathbf{l},\mathbf{l}') = \mathbf{l}\begin{bmatrix} 0 & -(333) & (233) & (133)-(223) & (222)-2(123) \\ (333) & 0 & -(133) & (123) & (113)-(122) \\ -(233) & (133) & 0 & -(113) & (112) \\ (223)-(113) & -(123) & (113) & 0 & -(111) \\ 2(123)-(222) & (122)-(113) & -(112) & (111) & 0 \end{bmatrix}\mathbf{l}'^t$$

etc., and

$$H = \begin{bmatrix} a_0 & 4a_1 & 6a_2-\mathbf{2(11)} & 4a_3-\mathbf{2(12)} & a_4-\mathbf{2(13)} \\ 4a_1 & 16a_2+\mathbf{4(11)} & 24a_3+\mathbf{2(12)} & 16a_4-\mathbf{2(22)}+\mathbf{4(13)} & 4a_5-\mathbf{2(23)} \\ 6a_2-\mathbf{2(11)} & 24a_3+\mathbf{2(12)} & 36a_4+\mathbf{4(22)}-\mathbf{4(13)} & 24a_5+\mathbf{2(23)} & 6a_6-\mathbf{2(33)} \\ 4a_3-\mathbf{2(12)} & 16a_4-\mathbf{2(22)}+\mathbf{4(13)} & 24a_5+\mathbf{4(23)} & 16a_6+\mathbf{4(33)} & 4a_7 \\ a_4-\mathbf{2(13)} & 4a_5-\mathbf{2(23)} & 6a_6-\mathbf{2(33)} & 4a_7 & a_8 \end{bmatrix}$$

The curve:

$$y^2 = \mathbf{x}H\mathbf{x}^t = \sum_0^8 \binom{8}{i} a_i x^i$$

where $\mathbf{x} = (1, x, x^2, x^3, x^4)$.

The definition of the (ij), for (x_i, y_i) with $i = 1,2,3$ three arbitrary points on the curve:

$$yy_i = \mathbf{x}H\mathbf{x}_i^t \quad i = 1,2,3$$

The two index symbols decompose into a five dimensional irreducible,

$$(11) \xrightarrow{\mathbf{e}} -2(12) \xrightarrow{\mathbf{e}/2} (22)+2(13) \xrightarrow{\mathbf{e}/\mathbf{3}} -2(23) \xrightarrow{\mathbf{e}/4} (33) \xrightarrow{\mathbf{e}} 0$$

$$0 \xleftarrow{\mathbf{f}} (11) \xleftarrow{\mathbf{f}/4} -2(12) \xleftarrow{\mathbf{f}/3} (22)+2(13) \xleftarrow{\mathbf{f}/\mathbf{2}} -2(23) \xleftarrow{\mathbf{f}} (33),$$

and a one dimensional irreducible,

$$(22)-4(13) \xrightarrow{\mathbf{e}} 0$$

$$0 \xleftarrow{\mathbf{f}} (22)-4(13)$$

8

These identities are obtained by calculation in the following manner:

- $yy_i - \mathbf{x}H\mathbf{x}_i^t = 0$ has a residue at $(x,y) = (\infty,\infty)$ leading to identities in x_i, y_i and (ij);
- Apply $y\partial_x = \mathbf{x}\cdot\nabla_{\mathbf{u}}$ to $yy_i - \mathbf{x}H\mathbf{x}_i^t = 0$. Residue at $(x,y) = (\infty,\infty)$ leads to identities in $x_i, y_i, (ij)$ and (ijk);
- Eliminate $x_i's, y_i's$ to obtain highest weight ($\in \ker \mathbf{f}$) identities in (ij) and (ijk).
- Generate full modules of identities by application of $\mathbf{e}$ to highest weight elements.

Amongst these identities one finds the matrix relation:

$$HA = 0$$

where H is the $(g+2)\times(g+2)$ matrix entering earlier and A is either the 4-vector of (ijk) entering in the genus two case or the 5×5 antisymmetric matrix of (ijk)'s in the genus three case.

In both cases the H matrix has rank three. (*So a reduction to quadratic normal form shows there are* $\frac{1}{2}g(g-1)$ *Kummer relations amongst the* (ij)) In genus two A is a rank one vector; in genus three it is a rank two matrix. In the latter case the vanishing 3×3 minors of A are a five dimensional module of (Plücker) relations:

$$\begin{aligned}
(113)(333)-(123)(233)+(223)(133)-(133)^2 &= 0\\
-(113)(233)-(112)(333)-(133)(222)+2(133)(123)+(233)(122) &= 0\\
(133)(122)-(133)(113)-(223)(122)+(223)(113) &\\
+(111)(333)+(123)(222)-2(123)^2 &= 0\\
-(233)(111)-(112)(133)+(112)(223)-(113)(222)+2(113)(123) &= 0\\
(133)(111)-(123)(112)+(122)(113)-(113)^2 &= 0
\end{aligned}$$

9. A conjecture for general genus

$$A(\mathbf{l}_1,\dots,\mathbf{l}_{g-1})A(\mathbf{k}_1,\dots,\mathbf{k}_{g-1}) = -\frac{1}{4}\det\begin{bmatrix} H & \mathbf{k}_1^t & \dots & \mathbf{k}_{g-1}^t \\ \mathbf{l}_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \mathbf{l}_{g-1} & 0 & \dots & 0 \end{bmatrix}$$

On the right hand side of the conjecture:

- $\mathbf{l}_i$ and $\mathbf{k}_i$ for $i = 1,\dots,g-1$ are $(g+2)$-vectors of arbitrary parameters which are a basis for a V_{g+2};
- H is the $(g+2)\times(g+2)$ matrix in the covariant definition : $yy_i = \mathbf{x}H\mathbf{x}_i^t$ for $i = 1,\dots,g$;
- The expression can be expanded as a quadratic form (*in the Plücker coordinates*) on the Grassmannian, $Gr(g-1, g+2)$.

On the left hand side of the conjecture:

- Each A is to be linear in the (ijk) which are a basis for the module Sym^3V_g;
- Each A is linear in the Plücker coordinates;
- Each A is to be invariant under $\mathfrak{s}l_2$.

In order for this to work we require an $\mathfrak{sl}_2$-module isomorphism,

$$Sym^3 V_g \cong \bigwedge^{g-1} V_{g+2}.$$

We are then able to pair up the irreducible components to create invariants.

For genus two: $Sym^3 V_2 \cong V_4 \cong \bigwedge^1 V_4$.

For genus three: $Sym^3 V_3 \cong V_7 \oplus V_3 \cong \bigwedge^2 V_5$.

For our immediate purposes however we require only that the parameters $\mathbf{l}_i$ etc., belong to the *Rational Normal Curve*, the embedding of $\mathbb{P}^1$ into $\mathbb{P}^{g+2}$:

$$(\eta, \zeta) \mapsto (\eta^{g+1}, \eta^g \zeta, \ldots, \zeta^{g+1}).$$

More generally, it *appears* that,

$$Sym^m V_g \cong \bigwedge^{g-1} V_{m+g-1}.$$

Representing a basis element in either set as a line of $m+g-1$ balls, $g-1$ of which are white and the remaining m black,

$$(\bullet \ldots \bullet \circ \bullet \ldots \bullet \circ \ldots\ldots\ldots \bullet \circ \bullet \ldots \bullet)$$

one sees that each module is $\binom{m+g-1}{g-1}$ dimensional. To exhibit the isomorphism we restrict to the normal curve, in which case the objects of $Sym^m V_g$ become linear expressions in the elementary symmetric polynomials, the objects of $\bigwedge^{g-1} V_{m+g-1}$ become, up to an invariant factor, Schur polynomials and the desired isomorphism is the Giambelli identity.

Here is an explicit example for $g = 3$, $m = 3$ (*We use square brackets for totally antisymmetric objects and round brackets for totally symmetric objects*):

$$\begin{aligned} \bigwedge^2 V_5 \ni [14] &= \begin{vmatrix} \eta_1^4 & \eta_2^4 \\ \eta_1\zeta_1^3 & \eta_2\zeta_2^3 \end{vmatrix} \\ &= (\eta_1\zeta_2 - \eta_2\zeta_1)\eta_1\eta_2((\eta_1\zeta_2 + \eta_2\zeta_1)^2 - \eta_1\eta_2\zeta_1\zeta_2) \\ [14] &\leftrightarrow (223) - (133) \end{aligned}$$

under the association ($\eta_1\zeta_2 - \eta_2\zeta_1$ *is* $\mathfrak{sl}_2$*-invariant*)

$$\begin{aligned} \zeta_1\zeta_2 &\leftrightarrow (1) \\ \zeta_1\eta_2 + \zeta_2\eta_1 &\leftrightarrow (2) \\ \eta_1\eta_2 &\leftrightarrow (3) \end{aligned}$$

The one remaining issue is to fix the normalisation constants for each pairing. This can be done by setting all the a_i to zero and using the usual Schur-Weyl limit.

10

In conclusion, we have tried to show that even very elementary representation theory is a powerful aid in the description of spaces of functions on curves and their Jacobians. It obviates *some* of the need for computing power. In particular we have shown that the genus one, two and three quadratic identities for three index $\wp$-functions of hyperelliptic curves are compactly written using such tools. Further there is geometrical shape to the identities:

- in genus three the three index $\wp$-functions give a map from the Jacobian into the Grassmannian variety, $Gr(2, 5)$;

- the genus three quadratic identities are parametrised by points in $Gr(2,5)$ arising from an embedding of the two-fold antisymmetric product of the rational normal curve with itself;
- an obvious conjecture exists to hyperelliptic curves of higher genus.

Further progress is being made in the following directions:

- A classical proof of the conjecture;
- The extension of these methods to more general classes of curve - certainly the "leading order" terms in those identities derived for non-hyperelliptic curves *do* obey the correct representation theory;
- Implications for the structure of the Θ-function;
- Covariance of the addition laws.

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